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INTERTEMPORAL GENERAL EQUILIBRIUM MODEL OF THE RUSSIAN ECONOMY BASED ON NATIONAL ACCOUNTS DEAGGREGATION

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Abstract. The paper describes a three-product intertemporal general equilibrium model of the Russian economy. The System Analysis of Evolving Economy (SAEE) approach to modelling economic processes developed in 1975 is described. Based on SAEE Intertemporal Equilibrium Model with Control of Capital (IEMCC) and three-product nonlinear deaggregation of Russian national accounts by consumption are described. The model agents of two types are introduced: mass (rational) and individual (scenario) agents. The former type includes Bank, Producer, Household, Owner, and Trader. The latter type includes State, Central Bank, Exporter, and Importer. The model is reduced to a boundary-value problem for a significantly nonlinear system of equations. These equations are derived from the initial hypothesis by means of the original economic modelling assistance system ECOMOD which is developed in the computer algebra system MAPLE. The model is adjusted with respect to official statistical quarterly data from 2004 till 2010. The calculation results are provided. The model proves the ability to model the Russian economy as a result of interactions of rational and individual macroeconomic agents.

1. Introduction

1.1. The Economy as an Object of Mathematical Modelling. Modern economies solve an enormously complex problem of managing the production of billions of goods and their distribution among billions of individuals and legal entities. For this reason the economy as a managing system has always been decentralized. Even the Government Planning Commission (Gosplan) of the USSR operated on at most hundreds of thousands of product names, whereas the real range of product varieties in the 1970s exceeded one hundred million. It was a disastrous mismatch of control and monitoring abilities to the rapidly increasing complexity of economic relations that, in our opinion, caused the failure of the idea of central planning.

Hence, in the modelling process the economy should be considered as a *complex system.* Such complex systems as the Universe, biosphere, a particular organism, a particular individual, natural language, the economy, are characterized not only by a multiplicity of components, but primarily by their *uniqueness* and, most importantly, by their *ability to perform endogenous qualitative changes.* Thus, when we study a complex system, we actually observe only a trajectory which is not necessary statistically significant and does not show the full potential of the system. Thus, we remain *beyond the empirical method*—the reason of success of natural sciences. Not surprisingly, the success in modelling of complex systems is still much more modest than the successes in modelling of, for example, technical systems.

For complex systems experiments and mass observation are impossible. Therefore, there is still no universal model of a complex system, from which the remaining models would follow as special cases (just like models of radio devices follow from the theoretical framework of electrodynamics).

Instead we deal with many models of the same system that differ in conceptual language and the way the system is contemplated, hence, they neglect irrelevant but not always small parameters [39]. While models of physical systems are used to explain the outcomes and predict the results of planned experiments, the models of complex systems are designed, first of all, *to replace the experiment* [19].

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1.2. The Tools of Economic Modelling. There is not only a set of models of the economy, but also a variety of methodological approaches to modelling.

First, the models differ in the object: *models of specific enterprises or markets*, *micro models* (models of typical enterprises or markets), *macro models* (models of economic systems).

It might seem that micro models should be derived from specific models by generalizing, and macro models from micro-models are derived by aggregation, but in practice this is rarely possible. The following discussion is focused on macro models.

Applied macro models may be broken down into the following groups according to the methods of construction.

Econometric models often rely primarily on the observed correlations between time series of economic indicators, instead of hypotheses about current relations in the economy. Once it is found that certain indicator time series can be expressed as a function of some other indicators with small and independent errors, one concludes that a pattern is detected. When the number of such relations is sufficient to determine all the quantities in terms of their past values, the resulting system of relations is formally able to predict the future values of indicators. This is an econometric model. Such models sometimes reach enormous sizes, containing hundreds of thousands of variables and relations.

An econometric model with about ten variables is currently used by the Ministry of Economic Development of Russia to forecast growth and inflation in order to justify the public budget draft. In addition, compact econometric models are widely used to organize or process raw data for models of other types and in various information systems. Practice shows that *an econometric model is unable to predict trend alteration*: once set for growth modelling, it will tend to predict growth. Econometric models are of little use for analytical calculations, that is, for addressing questions of the kind "What would have happened if another policy had been applied?".

Balance models emerged as a support method for economy planning. The main part of these models is the system of material balances for a set of products that encompass the entire economy. Planning procedures all over the world in 1950–1970s were based on the *Leontief model*. Nowadays a number of research groups apply the Leontief model to forecast the Russian economy [14, 42]. Presently balance models are often complemented by financial balances [8]. However, market mechanisms are typically not included in balance models, and the connection between the material and the financial parts is specified explicitly. As a result, a balance model with finance balances usually looks somewhat eclectic.

The advantage of balance models is that they are composed almost exclusively of the most robust economic balances and based on the data collected specifically for the model. The disadvantage is that the language of the balances does not reflect the relations between economic agents and specifics of their behaviour. For this reason the balance models are often unable to catch the actual problems of economic development. It is also worth remembering that the balance models describe at most several thousands and typically tens of products. "The products" in the balance models are aggregated, or combined into indexes of real goods by means of prices, exchange rates, payment flows, and accounting evaluations. Still, the primary and the most precise information in the economy is information about financial flows.

Theoretical models are constructions that formalize some descriptive economic theory. Contemporary economics textbooks are full of models of this type. These models are often used as a basis to justify economic policy. Careful analysis shows, however, that in all substantial theories the concepts slightly vary from chapter to chapter and from topic to topic. For this reason applications of the main clauses of the theory result not in a single model that corresponds to the theory, for instance, of John Maynard Keynes, but thousands of Keynesian models noncomparable with each other.

Simulation models originate from attempts to apply the technical systems modelling methods in economics. The basic technique of simulation modelling is system separation into blocks corresponding to the essential processes or objects. Then the system is described in terms of individual blocks. There is a difficulty in applying this method to the simulation of the economy. It lies in the fact that, in contrast to the technical system, which is made up of separate parts, the economy emerges in the process of self-organization, and division into parts is not unique.

The most popular were simulations of global dynamics [37,50,51]. However, the arbitrary assumptions and the gap with economic theory raised serious criticism of these models, and their negative predictions did not come true. Despite this, the method of *system dynamics* remained. System dynamics models that describe the practice of decision-making are now used in almost all major corporations [40]. Their real benefit is unclear, because these models often play illustrative and advertising roles. Also, they are often difficult to distinguish from computer games.

Synergistic models. Recently considerable interest has been given to, in a sense, the inverse trend of constructing relatively simple models based on analogies with the well-studied physical and biological processes. The basis of the analogy is formed out of qualitative phenomena inherent in dynamic systems of a certain mathematical type. These models are called synergistic, and recently—econophysics models [18, 33, 38, 41]. Sometimes they do offer unexpected effects and connections, but for now, we believe, they are not quite reliable and practical.

Computable General Equilibrium Models (*CGE*) since the 1990s have become the main tool in worldwide practice for economic forecasting [45, 47], because it appears that accounting for some technological constraints (balance models), extrapolating past trends (econometric models), and directly imposing external constraints (system dynamics models) is not enough to adequately describe the modern economy.

CGE models go back to the dynamic version of the general economic equilibrium model by K. J. Arrow and Debreu [46]. General equilibrium models, especially the dynamic ones, are a complex bunch of nonlinear optimization problems (see Sec. 3). In CGE models these dynamic relations are usually replaced by certain phenomenological assumptions.

1.3. System Analysis of Evolving Economy (SAEE). This stream of research was established in 1975 in the Computing Centre of the USSR Academy of Sciences (later the Computing Centre of Russian Academy of Sciences) by the Academician A. A. Petrov and one of the authors of this paper I. G. Pospelov (presently corr. member of RAS). It was challenged to synthesize the methodology of mathematical modelling of complex systems (developed in natural sciences) with the achievements of modern economic theory [17, 20]. The system analysis models of an evolving economy are in some sense close to CGE models, but pay more attention to the specifics of current economic relations, and started about 15 years prior to the onset of CGE models.

In 1988 a model was developed that reproduced the main qualitative properties of the evolution of the planned economy. Therefore, by the time of the economic reforms in USSR and later in Russia we already had an approach for analysis of the changes in the economy. In particular, two years before the reform of 1992, our group was able to predict correctly the short-term outcomes. Within the next few years were built: a model of the economy of the period of high inflation in 1992–1995; a model of the economy during the 1995–1998 "financial stabilization" that predicted the financial crisis of 1998; and a model to estimate the economic outlook after the 1998 crisis that was based on a system of hypotheses about the nature of the economic relations that evolved in the corresponding period in Russia [22, 23].

The models allowed us to understand the internal logic of the development of the economic processes that were hidden behind the visible, often seemingly paradoxical picture of the economic phenomena that did not fit in the known theoretical schemes. The experience of application of these models showed that they serve as a reliable tool for macroeconomic analysis and forecast of the effects of macroeconomic decisions *in case the existing relations maintain.* One can say, a complete "chronicle " of Russian economic reforms was written in the language of mathematical models [7, 21, 23].

The main difficulty of modelling the Soviet and Russian economy in the period of 1986–2004 was that each successive model had to be created from scratch, starting with a systematic analysis of changed economic relations. Development of a new model is a very time-consuming process; it takes about a year of work of a team of qualified professionals. But even this is not the main point. New economic relations are often described by brand new variables and expressions, and require use of new mathematical methods. Summarizing, one cannot say that after thirty years of research evolution of the Soviet and Russian economy the *system* of models has been created. The above models are as difficult to compare as the models created by different research groups. This situation is, unfortunately, typical for modelling complex systems [26].

In 2004 we changed our approach to modelling; in particular we abandoned the simplification of dynamic relations typical for CGE models and earlier SAEE models. We turned to the theoretically more consistent, but technically and conceptually much more complex, design of *intertemporal equilibrium with control of capital* (see Sec. 3). The first applied model, which implemented this theoretical concept, was a single-product model of intertemporal equilibrium of the Russian economy. It was developed for the Federal Agency of Taxation to estimate the size of the shadow economy based on the nontax data [7]. The model considered below was developed in a project funded by the Central Bank, and is a direct development of [7]. The basic and significant difference is that this model describes the turnover of three products (Sec. 4).

2. The Structure of the Model

2.1. The System of Material and Financial Balances. SAEE models describe economic dynamics as a result of interaction of some *economic agents*. It is based on a complete system of material and financial balances. Material balances correspond to material goods: resources and products (commodities and services). Completeness of material balances means that for each of the considered material goods the balances describe its transfers from agent to agent, starting from production (or appearance from the external environment in case of resources) up to the terminal use. In general, the material balance equation is as follows (Fig. 2.1.1):

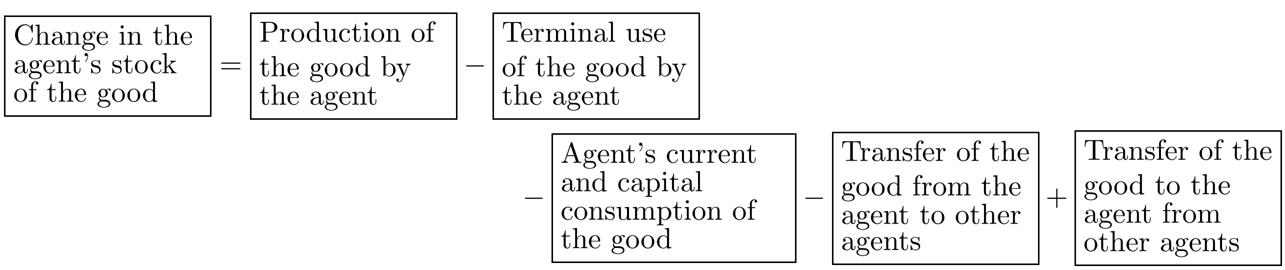

Fig. 2.1.1. The general form of the material balance.

Note that the right hand side has different **flows** of the good. The last two terms on the right hand side of this equation express the most important property of *additivity of the goods*: one agent loses what he gives the other agents. Additivity allows us to aggregate the balances: *for any set of agents the total stock is described by the balance of the same type*[1] [28]. For this reason the economic statistics is based on principle that such a balance might be written for each specific product (see Sec. 4.1). However, in reality it is of course impossible. The lists of goods are boundless and constantly change in composition. Therefore, in fact, both statistics and models use the **aggregated** products measured mainly in terms of cash flows. In our model we describe the balances of **homogeneous labor**, and three aggregate products: **export**, **import** and **"internal" product**. The original method of calculation of these units, as described in Sec. 4, is the most important feature of the model.

The intricate system of material balances requires aggregation of information about countless heterogeneous material goods. This is necessary, in the first place, not for the researcher of the economy, but for the person who lives in it. The information in the economy is aggregated and transferred by means

[1]In the future one probably will have to do without the system of material balances, since two classes of *nonadditive* goods are playing an increasing role in the economy: public goods and information. *Public goods* (order, justice, security, environmental comfort, etc.) are not distributed among agents. They, ideally, belong to everybody or to nobody. *Information* does not add up in a standard way. If one agent tells news to another agent, the former does not forget this piece of news and the latter does not benefit if the latter hears that same news again.

of *money*. In a developed economy a counter flow of cash payments corresponds to each systematically repeating flow of goods (Fig. 2.1.2).

[Payments flow between agents for the transfer of the good] = [The good price] × [The volume of transferred good between the agents]

Fig. 2.1.2. Connection between monetary and material flows.

Here it is assumed that *prices* are independent of pairs of agents that exchange (see Sec. 3.1). The stocks (residuals) of agents' cash are additive. Presently, since there is no natural money emission, these stocks satisfy the financial balance [28] shown in Fig. 2.1.3:

[Change in agent's stock of money] = [Surplus of payments for goods for all other agents[2]]
\+ [Surplus of transfers for other agents] + [Total increment in liabilities to other agents] − [Total increment in liabilities from other agents]

Fig. 2.1.3. The general form of financial balance.

In order to avoid negative values and make the accounting consistent with the algebra of the balance equations, it is necessary to treat liabilities (debt) as negative stocks. Because all liabilities are someone else's claims (assets), if we add the financial balances over all agents, we find that the sum of agents' money stocks does not increase with time (the flows of money are complete). This implies that some agents, namely issuers, must have negative money stocks, and the money of other agents is issuers' debt [29].

Figure 2.1.3 shows the dynamic balance of the agent's money. One can reduce this equation to normal mathematical form if one denotes increment in liabilities and claims on the right hand side by new money flows. As a result one obtains equations for change in claims and liabilities. This will give a system of ordinary accounting balance equations which describe the changes in the financial instruments by means of *double postings*.[3] If one takes each of these instruments and sums over the agents, one receives the identity equation of the total assets to the total liabilities. Therefore, money appears to be the liability of the Central bank. This is completeness of the financial balances in the system.

There are many financial instruments (more than 1000 in a standard bank balance sheet [5]), and they also have to be aggregated. They are easier to aggregate than material goods, because financial instruments are measured in money. Our model includes a complete set of balances for seven financial instruments: **cash balances**, **current accounts**, **correspondent accounts in the Central Bank**, **bank loans**, **bank deposits**, **net of bank deposits in the Central Bank**, and **foreign currency**.[4]

2.2. The Agents and Interactions: the Canonical Form of the Model. We describe the dynamics of the Russian economy as a result of interaction of the following nine agents:

- **Producer** represents all nonfinancial commercial enterprises;
- **Bank** represents all financial commercial organizations;
- **Household** represents individuals that act as customers and employees;
- **Owner** represents individuals and legal entities that control the capital allocation between sectors of the national economy and abroad;

[2]Including financial services—received and paid interest surplus.

[3]The second posting for payments and transfers changes the capital account.

[4]When modelling the national economy the issuer of the foreign currency appears to be outside of the system. Therefore, the balances in the model of the Russian economy describe currency as a material good, not as a financial instrument.

- **State** represents agencies that implements budget policy and determines the parameters of economic policies (tax rates, reserves etc.);
- **Central Bank** is an issuer, the collector of reserves and the clearing center for commercial banks;
- **Merchant** is a mediator on the domestic market, performs aggregation of the model products described in Sec. 4;
- **Exporter** implements mediation in export of products and raises foreign investment;
- **Importer** mediates in import of products.

The main function of each economic agent in the model is to determine material and financial flows that are in "competence" of the agent. The complete formal description of their activities is provided in Sec. 6.

The idea of economic equilibrium is that each agent offers a plan of flows (demand or supply). The plan is conditional: it depends on the values of particular **informational variables** (prices, interest rates, exchange rates), which deliver information about the state of the whole system to the agent. The plans that are valid under the existing economic relations are described by technological (internal) and institutional (external) constraints. The latter contain the information variables. The simplest example of institutional constraints is the relation between the flow of money and the flow of goods at a specified price that serves as an information variable. It is shown in Fig. 2.1.2.

Interactions bring the plans of agents into agreement. Their equilibrium values are chosen to satisfy the balance relations in the model and are determined by the information variables. Thereby we obtain the scheme shown in Fig. 2.2.1, which is named the **canonical form of the model** [7, 15].

This scheme means that the equations of the model in the canonical form split into blocks of two types: the description of agents behavior (EA—Economic Agent) and the description of agents interaction (IA). In general, interaction does not always involve agents' plans, but sometimes it involves exchange of information. One may write all SAEE models in the canonical form, including the model of the planned economy,[5] all CGE models, the "theoretical" models and most simulation models, but not all econometric, balance, and synergy models.

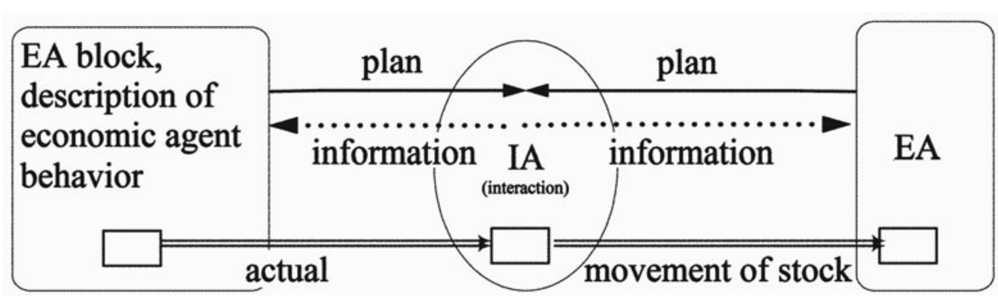

Fig. 2.2.1. Scheme of the model in the canonical form.

The main problem when constructing any model in the canonical form is the choice of aggregate additive variables. For them one writes a complete system of material and financial balances and records the institutional constraints that describe economic relations in terms of these variables. It is the form and the set of the constraints that expresses the difference between competitive and monopoly markets [7], the difference between shareholder equity ownership and partnership [24], the use of monetary aggregates [28], and the channels of shadow turnover [7].

It is worth emphasizing that the models of SAEE and CGE differ in the system of institutional constraints. If the former use mostly the textbook institutional constraints, SAEE models try to reflect the specifics of the economy under consideration.

[5]Instead of prices, in this model the information variables were the levels of shortages and the quality of the products [22].

2.3. Rational Behavior: Aggregate and Individual Agents. Institutional and technological constraints usually provide the agent with sufficient freedom to choose the plan. This raises an important question of the way to describe agents behavior. From the point of view of the agent's problem the agents listed in the beginning of the Sec. 2.2 split into two different types. The first are Producer, Bank, Household, and Owner. These are the **macroagents**, that describe the combined effect of a large number of relatively independent individuals or organizations performing similar roles in the economy. The second type are State and Central Bank. These are **individual agents**, which represent specific organizations with will and that report about their actions. The agents Merchant, Exporter and Importer are actually created to simplify the structure of the model. We formally consider the Merchant as a macroagent and the Exporter with the Importer as individual agents.

The experience of operations research and the solutions of the design and planning problems suggest that the mass agents should be described as statistical ensembles, and individual agents—as rational planners. However, one may notice that the economic theory and models based on the economic theory assume the opposite. For example, a typical textbook model [32] studies the impact of taxes imposed by the state on the equilibrium of the market, where demand is determined by maximization of the household's utility, and supply—by maximization of the firm's profits. In this paper the macroagents, i.e., firms and households, are optimizers, and the individual agent, i.e., the state, is described by possible actions without any optimization.

We apply the traditional approach of economic theory, which is based on the world experience in economic modelling. Repeated attempts to describe the macroagents alternatively, for example statistically, did not make any real progress. We, nevertheless, have doubts in the traditional rationale for the economic theory approach. The existing theory deals with "representative agents" that are characterized by given settled goals. Microeconomic studies sometimes try to reveal differences in interests, but the macroeconomic models almost always deal with the single representative agent of each type (consumer, producer, trader, etc.).

We believe that the idea of absolutely self-sufficient "representative individuals" contradicts the fact that people and organizations interact with each other. Economic subjects that play similar roles within large groups may compete and imitate each other [31]. *As a result, the collective behavior of the group appears to be simpler and more consistent than the behavior of its members* [30, 48], and can be simply described as a desire to maximize consumption, income, wealth etc., which can be confirmed by direct measurements. The quantitative examples of such "collective rationality" are given in Sec. 4.3. (For more details, see [16].)

To summarize, in the model we assume the behavior of macroagents to be **rational**, that is, optimizing some criterion, and the behavior of individual agents to be described by possible **scenarios** of economic policy.

3. Intertemporal Equilibrium with Control of Capital

3.1. A Typical Problem of a Macroagent and the Principle of Rational Expectations. The **state** of an agent a at the moment of time[6] t in the model is characterized by agents' stock of material goods and financial instruments (see Figs. 2.1.1 and 2.1.3). We denote the vector of these stocks, except for the balance of money, by $\mathbf{x}_a(t)$; the stock (balance) of money of the agent a by $A_a(t)$; and the vector of flows that change the stocks $\mathbf{x}_a(t)$ by $\mathbf{y}_a(t)$.

We now make two assumptions typical for most dynamic macroeconomic models.

- In the institutional constraints (see, for example, Fig. 2.1.2), the agent perceives the prices, the exchange and interest rates as informational variables, which the agent cannot control (a price taking agent).

[6]For the reasons set out in Sec. 5, we formulate and analyze the model in **continuous** time and then proceed to discrete time with the step of the data tick in order to make calculations.

- The interests (goals) of the agent are associated with a single quantity of income (the **useful flow**) $u_a(t)$: dividends in case of Producer and Bank, consumption expenditure in case of Household, net increase in foreign assets for Owner.[7]

Under these assumptions, when we reduce the financial balances to the normal form of dynamic equations and express the cash flows from the institutional constraints-equations, similar to that shown in Fig. 2.1.2, we often[8] come to the description of agent capabilities as the system of the following restrictions on the **agent's planned variables** $A_a(t)$, $\mathbf{x}_a(t)$, $\mathbf{y}_a(t)$, $u_a(t)$:

$$\frac{d}{dt}A_a(t) = \mathbf{r}(t)\mathbf{x}_a(t) + \mathbf{p}(t)\mathbf{y}_a(t) - u_a(t), \tag{3.1}$$

$$\frac{d}{dt}\mathbf{x}_a(t) = -\mathbf{R}(t)\mathbf{x}_a(t) - \mathbf{y}_a(t), \tag{3.2}$$

$$\mathbf{g}\big(t, A_a(t), \mathbf{x}_a(t), \mathbf{y}_a(t)\big) \geq 0. \tag{3.3}$$

Here the components of the vector $\mathbf{r}(t)$ are interpreted as interest rates; the components of $\mathbf{p}(t)$—as prices or rates, possibly adjusted for tariffs and taxes on the assets $\mathbf{y}_a(t)$. The components of the "technology" matrix $\mathbf{R}(t)$ describe such flows as depreciation or debt reimbursement, and inequalities (3.3) describe the remaining technological and institutional constraints such as nonnegativity of stocks. Another terminal constraint will be added in Sec. 3.4.

As is often assumed, the agent's goal is maximization of the expected utility [32] from the flow $u_a(t)$:

$$\int_{t_0}^{T} U\big(t, u_a(t)\big)\, dt \to \max_{\substack{A_a(\cdot)\mathbf{x}_a(\cdot),\\ \mathbf{y}_a, u_a(\cdot)}}. \tag{3.4}$$

Here $U(\cdot,\cdot)$ is the utility function with explicit dependence on time. We use this form in order to take into account both the time preference (discounting) and to bring the cash flow $u_a(t)$ to "the real terms" (see Sec. 4). For the initial moment of time t_0 the values of stocks are given and in practice t_0 is chosen in the past in order to compare the part of the model trajectory with the known statistical data. The planning horizon T is chosen so as to embrace the forecast period.

Thus, formally rational behavior of a macroagent in the model is described as a solution of the optimal control problem—maximizing the functional (3.4) for $t \in [t_0, T]$ over the absolutely continuous $A^a(t)$, $\mathbf{x}^a(t)$ and measurable $\mathbf{y}^a(t)$, $u_a(t)$ almost everywhere in $[t_0, T]$, that satisfy (3.1)–(3.3) given the initial conditions $A(t_0)$ and $\mathbf{x}(t_0)$ and the measurable *bounded* functions $\mathbf{r}(t)$, $\mathbf{p}(t)$, and $\mathbf{R}(t)$.

In our model as well as in the majority of economic models

$$U(t,\cdot) \text{ and } \mathbf{g}(t,\cdot,\cdot,\cdot) \text{ are } \textit{smooth} \text{ and } \textit{concave} \text{ for } t \in [t_0, T]. \tag{3.5}$$

In addition, we make natural assumptions that the utility is monotone, and an increase in cash stock does not tighten the restrictions on the agent's actions, all other things being equal:

$$\frac{\partial U}{\partial u} > 0, \quad \frac{\partial \mathbf{g}}{\partial \mathbf{A}} \geq 0. \tag{3.6}$$

A substantial difficulty with the problem (3.4), (3.1)–(3.3) is that a macroagent plans its actions for the future and, therefore, has to predict the future changes in market conditions: $\mathbf{r}(t)$, $\mathbf{p}(t)$, and $\mathbf{R}(t)$. There is a paradox: *we create a model to give a forecast of the market conditions, but to build the model we must know how agents predict the market situation.* The radical solution to this paradox, which we fully accept, is the **rational expectations principle** [46]: agents can make a prediction to the same extent as the author of the model.

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[7]Such an asset as "money" is formally distinguished from other financial instruments because its balance includes the flow of useful spending. For different agents a variety of tools may serve the role of money (see Sec. 6).

[8]The descriptions of agents in Sec. 6 somewhat differ from the typical problem, but this only affects interpretation of the solution, not the approach to description of the macroagent's behavior. For details, see [7].

Although the principle of rational expectations creates reasonable doubts, since it implies that the modelled agents "know everything in advance[9]," we took a chance to apply it to modelling the Russian economy. As a result, we have succeeded more than with the phenomenological simplified description of the behavior of agents characteristic of CGE models and the earlier SAEE models. In Sec. 7.4 we discuss several possible reasons for this unexpected success.

3.2. Sufficient Optimality Conditions. Since the external factors $\mathbf{r}(t)$, $\mathbf{p}(t)$, and $\mathbf{R}(t)$ are determined mainly from interactions of agents, it is impossible to solve an agent's problem (3.4), (3.1)–(3.3), say, numerically, before the whole model is built. Therefore we substitute the description of the behavior of a macroagent in the form of the optimization problem with a system of optimality conditions. The problem (3.4), (3.1)–(3.3), however, belongs to a class of very complicated nonautonomous problems with mixed constraints [1]. For arbitrary functions $\mathbf{r}(t)$, $\mathbf{p}(t)$, and $\mathbf{R}(t)$, the necessary optimality conditions are immensely complex. But we have to solve this problem not for arbitrary functions, but for the equilibrium functions, that is, the functions $\mathbf{r}(t)$, $\mathbf{p}(t)$, and $\mathbf{R}(t)$ that somehow correspond to the main problem.

Therefore, we believe it is possible to limit the analysis to simple **sufficient optimality conditions**. It is known [1, 27] that the trajectory of the **direct variables** $A^a(t)$, $\mathbf{x}^a(t)$, $\mathbf{y}^a(t)$, and $u_a(t)$ is optimal in the problem (3.4), (3.1)–(3.3), if there are absolutely continuous **dual variables** $\xi_a(t)$, $\tilde{\psi}_a(t)$ and measurable nonnegative dual variables $\varphi_a(t) \geq 0$, which, together with $A^a(t)$, $\mathbf{x}^a(t)$, $\mathbf{y}^a(t)$, and $u_a(t)$, form a saddle point of the Lagrange functional:

$$\int\limits_{t_0}^{T} \biggl[U\bigl(t, u_a(t)\bigr) + \xi_a(t) \biggl(\mathbf{r}(t)\mathbf{x}_a(t) + \mathbf{p}(t)\mathbf{y}_a(t) - u_a(t) - \frac{d}{dt} A_a(t) \biggr)$$

$$+ \tilde{\psi}_a(t) \biggl(\mathbf{R}(t)\mathbf{x}_a(r) - \mathbf{y}_a(t) - \frac{d}{dt}\mathbf{x}_a(t) \biggr) + \tilde{\varphi}_a(t)\mathbf{g}\bigl(t, A_a(t), \mathbf{x}_a(t), \mathbf{y}_a(t)\bigr) \biggr] dt \to \min_{\substack{\xi_a(\cdot)\psi_a(\cdot),\\ \varphi_a(\cdot)\geq 0}} \max_{\substack{A_a(\cdot),\mathbf{x}_a(\cdot),\\ \mathbf{y}_a(\cdot),u_a(\cdot)}} . \tag{3.7}$$

Minimization over $\xi_a(t)$ and $\psi_a(t)$ gives the initial system of equalities

$$\frac{d}{dt} A_a(t) = \mathbf{r}(t)\mathbf{x}_a(t) + \mathbf{p}(t)\mathbf{y}_a(t) - u_a(t), \tag{3.8}$$

$$\frac{d}{dt} \mathbf{x}_a(t) = -\mathbf{R}(t)\mathbf{x}_a(t) - \mathbf{y}_a(t). \tag{3.9}$$

Minimization over $\tilde{\varphi}_a(t) \geq 0$ gives the **complementary slackness conditions**

$$\mathbf{g}\bigl(t, A_a(t), \mathbf{x}_a(t), \mathbf{y}(t)\bigr) \geq 0, \quad \tilde{\varphi}_a(t) \geq 0, \quad \tilde{\varphi}_a(t)\mathbf{g}\bigl(t, A_a, (t), \mathbf{x}_a(t), \mathbf{y}_a(t)\bigr) = 0. \tag{3.10}$$

If one integrates the functional (3.7) by parts it appears that due to (3.5) it is concave in $A_a(t)$, $\mathbf{x}_a(t)$, $\mathbf{y}_a(t)$, and $u_a(t)$. Thus the maximum with respect to these variables over the interval (t_0, T) is given by the following relations:

$$\frac{\partial}{\partial u} U\bigl(t, u_a(t)\bigr) = \xi_a(t), \tag{3.11}$$

$$\frac{d}{dt}\xi_a(t) + \tilde{\varphi}_a(t)\frac{\partial \mathbf{g}}{\partial A}\bigl(t, A_a, (t)\mathbf{x}_a(t), \mathbf{y}_a(t)\bigr) = 0, \tag{3.12}$$

$$\frac{d}{dt}\tilde{\psi}_a(t) - \tilde{\psi}_a(t)\mathbf{R}(t) + \xi_a(t)\mathbf{r}(t) + \tilde{\varphi}_a(t)\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\bigl(t, A_a, (t), \mathbf{x}_a(t), \mathbf{y}_a(t)\bigr) = 0, \tag{3.13}$$

$$\xi_a(t)\mathbf{p}(t) - \tilde{\psi}_a(t) + \tilde{\varphi}_a(t)\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\bigl(t, A_a, (t), \mathbf{x}_a(t), \mathbf{y}_a(t)\bigr) = 0. \tag{3.14}$$

[9] Any alternative to this principle requires an explicit description of how the economy evolves, and separately—what the agents think about the economy. But is it possible for the author "to think for all the agents better than they do," and has the author the right to consider the others being unable to carry out the same reasoning? The principle of rational expectations makes the author equal to persons whose behavior the author studies.

The variation with respect to the terminal values $A_a(T)$ and $\mathbf{x}_a(T)$ is discussed in Sec. 3.4.

The most difficult part of the analysis is the complementary slackness conditions (3.10). In the end, we solve them partially, that is, either reduce them to one of the equations $g'\big(t, A_a(t), \mathbf{x}_a(t), \mathbf{y}_a(t)\big) = 0$ and $\tilde{\varphi}'_a(t) = 0$, or "smooth" (regularize) them. A concrete example is given in Sec. 6.3.2.

3.3. Intergal of the Capital in the Homogeneous Problem of a Macroagent. Suppose now that the macroagents' problem is *homogeneous* (scale-invariant), that is, the absolute size of the system controlled by the macroagent is not essential: if one doubles the initial values of stocks, then doubled planned flows and stocks are still admissible. It is easy to see that the equality constraints (3.8) and (3.9) satisfy this property. Therefore, for the whole problem to be homogeneous it is sufficient to require the function $\mathbf{g}(t, \cdot, \cdot, \cdot)$ to be linearly homogeneous. Then, by Euler's theorem

$$\mathbf{g} \equiv A\,\frac{\partial \mathbf{g}}{\partial A} + \mathbf{x}\frac{\partial \mathbf{g}}{\partial \mathbf{x}} + \mathbf{y}\frac{\partial \mathbf{g}}{\partial \mathbf{y}}. \tag{3.15}$$

The assumption of homogeneity (constant returns to scale) is more or less common to all popular economic models. It is sufficient to say that most known economic laws (for example, the famous R. Solow's "golden rule" [32]) are obtained from the analysis of the balanced growth in the simple models, and balanced growth is realized provided homogeneity.

Homogeneity of the real economy is questionable: the natural resources are limited, the growth rate of production is not equal to that of population, the need for any specific good is saturable—all this contradicts the notion of uniformity and balanced growth. However, during the 200 years of its existence, the industrial economy managed to continue growing exponentially, overcoming (so far) all external constraints: the land constraints in the early 19th century, labor constraints in the early 20th century, energy constraints in the 1980s. The forecasts based on the limited capacity (including the views of Malthus and Forrester) dramatically failed. Now, when we are speaking about the economic processes, we are only interested in pace and proportions[10]; thus the assumption of homogeneity is justified.

Homogeneity means a certain symmetry of the problem. According to Noether's theorem [1], each symmetry of the variational problem corresponds to the first integral of the field of extremals (conservation law). In our case the symmetry is somewhat disrupted by the presence of a term $U(t, u)$. Thus, instead of pure conservation of (3.8)–(3.14) the condition (3.15), as might be easily verified, gives a simple differential equation

$$\frac{d}{dt}\big(\xi_a(t)A_a(t) + \tilde{\psi}_a(t)\mathbf{x}_a(t)\big) + \xi_a(t)\,u_a(t) = 0. \tag{3.16}$$

To understand its economic meaning, one may note that by (3.6) and (3.11) the dual variable to the balance of money $\xi_a(t)$ is positive and decreases with time:

$$\xi_a(t) \geq 0, \quad \rho_a(t) \triangleq -\frac{1}{\xi_a(t)}\,\frac{d}{dt}\,\xi_a(t) \geq 0, \tag{3.17}$$

so that we may introduce a variable

$$\Omega_a(t) \triangleq A_a(t) + \frac{\tilde{\psi}_a(t)}{\xi_a(t)}\,\mathbf{x}_a(t), \tag{3.18}$$

which due to (3.16) on the optimal trajectory (3.8)–(3.14) with (3.15) satisfies the equation

$$\frac{d}{dt}\Omega_a(t) = \rho_a(t)\Omega_a(t) - u_a(t). \tag{3.19}$$

It is well known [1] that dual variables $\xi_a(t)$ and $\tilde{\psi}_a(t)$ show, in linear approximation, an increase in the optimal value of the functional (3.4) per unit increase in money stock $A_a(t)$ at moment t or in asset component $\mathbf{x}_a(t)$ corresponding to component $\tilde{\psi}_a(t)$.

[10] By how many times a certain indicator has changed. On the contrary, when we speak about physical life, we are often interested in speed (by how much has it changed). This means that in physics the absolute size is essential, and in economics it is not.

Thus, $\xi_a(t)$ and $\tilde{\psi}_a(t)$ from the agent's point of view represent objective prices of one's own money and assets in utility units. Relations

$$\psi_a(t) \triangleq \frac{\tilde{\psi}_a(t)}{\xi_a(t)} \tag{3.20}$$

determine accordingly "true" money prices of assets, while $\Omega_a(t)$ determine the theoretical value of **net assets**[11] or the agent's **own capital** [32].

Equation (3.19) shows that the agent's capital grows with the *return rate* $\rho_a(t)$ and shrinks due to utility expenses $u_a(t)$. The return rate is a nontrivial convolution of the current and expected real return rates of individual financial instruments and individual production processes in which the agent participates. Its form differs between agents (see Sec. 6).

If $u_a(t)$ grows at a sufficient rate and the capital remains positive, it follows from (3.18)–(3.20) that net assets are equal to expected income, discounted with net present value:

$$\Omega_a(t) = A_a(t) + \psi_a(t)\mathbf{x}_a(t) = \int_t^\infty u_a(\tau)\, e^{\int_t^\tau \rho_a(s)\,ds}.$$

Thus, in a homogeneous model of the agent, both *estimations of firm value*—by assets accumulated in the past and by income expected in the future—are identical, if assets prices and discounting coefficient are to be calculated from conditions of optimality of the agent's behavior.

The value $\rho_a(t)\Omega_a(t)$ is obviously interpreted as the agent's **balance profit** [32]. On the optimal trajectory (3.8)–(3.14) it is expressed in the form

$$\rho_a(t)\Omega_a(t) = \mathbf{r}(t)\mathbf{x}_a(t) + \big(\mathbf{p}(t) - \psi_a(t)\big)\mathbf{y}_a(t) + \left(\frac{d}{dt}\,\psi_a(t)\right)\mathbf{x}_a(t) - \psi_a(t)\,\mathbf{R}(t)\mathbf{x}_a(t). \tag{3.21}$$

Separate summands of this expression have a natural economics interpretation. The value $\mathbf{r}(t)\mathbf{x}_a(t)$ represents the profit from financial operations (the difference between interest paid and received). The value $\big(\mathbf{p}(t)-\psi_a(t)\big)\mathbf{y}_a(t)$ represents profit from the sale of produced goods (the margin between the selling price and the "true cost" $\psi_a(t)$). Finally, the value $\left(\frac{d}{dt}\,\psi_a(t)\right)\mathbf{x}_a(t)$ represents the **profit from assets reevaluation**, and the value $\psi_a(t)\mathbf{R}(t)\mathbf{x}_a(t)$ represents depreciation deductions. We stress that expression (3.21) for balance profit is not postulated but derived. We interpret the result in the following way: book-keeping includes profit from reevaluation (which is not in fact received from anyone) into income, and depreciation deductions (which is not paid to anyone) into expenditures, to make the evaluation of net assets based on net factual incomes and expenditures closer to the "true" evaluation based on the agent's theoretical value of capital (3.18).

Note that in the model one can carry out an analogous calculation not only for the macroagents Producer and Bank but also for Household, which is never done in book-keeping. Consumer expenses serve as utility expenses $u_a(t)$ for a Household, and savings interest serves as income from financial operations.

It is striking that all mentioned properties of agent's capital do not demand any additional assumptions. They follow from conditions of optimality and independence of scale the same way as, in physics, energy conservation follows from the principle of least action and the independence of the motion of a physical system follows from calendar time. Homogeneity can be somewhat broken in the description of specific agents, and this adds additional terms to the right hand side of (3.19). They can be interpreted as indirect income, lost profit, or monopolistic rent.

In the construction of intertemporal equilibrium with control of capital the $\Omega_a(t)$ variable is used to impose terminal condition (Sec. 3.4) and to describe interaction between Owner and his or her firm (Sec. 3.5).

[11] Liabilities make negative contribution to value (3.18). If we describe liability as negative stock, then corresponding component $\psi_a(t)$ will be positive. If by contrast, as in Sec. 7, we describe liability as an instrument, that increases its stock with expenses and decreases with profits, then the corresponding component $\psi_a(t)$ appears to be positive.

3.4. Terminal Condition and Turnpike Property of Agent's Problem. The system of optimality conditions (3.8)–(3.14), as already mentioned, is not complete, because the right end of the trajectory was not varied yet. If no conditions are imposed on the right end, then, as a rule, variation over terminal values produces relations incompatible with (3.8)–(3.14).[12] The substantive reason for this is that acting for his own benefit the agent will certainly try at the last minute to borrow a loan that the agent will not have to return. A problem statement with infinite time horizon often used in economic modelling does not help the situation because in problems with financial planning, it is usually possible in principle for an agent to make a pyramid scheme, and to exclude it one needs to put additional highly intricate conditions (no Ponzi game condition [53]).

For a new solution to this old problem it appeared possible to use the notion of capital discussed above. Specifically, we propose to impose *the single terminal condition of capital growth by a certain factor.*[13] With this condition, the agent's capital is positive on optimal trajectory and this replaces the "no Ponzi game" condition.

Solution with such a terminal condition is not unique however, since the form of capital contains dual variables not known in advance. This is unimportant though: it is sufficient to limit the dual variables on the right end in some way. The point is that there is a characteristic property for optimization of economic processes that is unknown in engineering and physics. It is the turnpike property [46, turnpike theorem]: *influence on the current optimal solution of future targets and external influences decays exponentially with distance from the present.* In other words, the economic system has "universally optimal" trajectories, which give acceptable results at different distant future implementations and different terminal conditions. No wonder several Nobel Prizes were awarded for the study of this remarkable property. It offers hope that our model calculations are quite reliable and independent of knowledge of the details of the future.

From a mathematical point of view, the qualitative difference of optimization problems of physical and economic background is as follows. Dynamical equations (3.8), (3.9), (3.12), and (3.13), taking into account (3.10), (3.11), and (3.14), just as in physics, form a *Hamiltonian system* [1]

$$\frac{d}{dt}\,\mathbf{q}(t) = \frac{\partial H}{\partial \pi}\big(t, \pi(t), \mathbf{q}(t)\big), \quad \frac{d}{dt}\,\pi(t) = -\frac{\partial H}{\partial \mathbf{q}}\big(t, \pi(t), \mathbf{q}(t)\big), \tag{3.22}$$

where $\mathbf{q} = \langle A_a, \mathbf{x}_a\rangle$, $\pi(t) = \langle \xi_a, \tilde{\psi}_a\rangle$, and the Hamiltonian H is defined as

$$H(t, \langle \xi_a, \tilde{\psi}_a\rangle, \langle A_a, \mathbf{x}_a\rangle) \\ = \min_{\varphi_a \geq 0}\, \max_{\mathbf{y}_a, u_a} \big[U(t, u_a) + \xi_a(\mathbf{r}(t)\mathbf{x}_a + \mathbf{p}(t)\,\mathbf{y}_a - u_a) + \tilde{\psi}_a(-\mathbf{R}(t)\,\mathbf{x}_a - \mathbf{y}_a) + \tilde{\varphi}_a(t)\mathbf{g}(t, A_a, \mathbf{x}_a, \mathbf{y}_a)\big].$$

Let us assume for simplicity that $\partial H/\partial t = 0$. Then movement of the system (3.22) takes place on surfaces of constant energy $H = \text{const}$. For Hamiltonians of physical origin it is typical to have minima and, respectively, to move along closed surfaces around them. There are countless neutrally stable oscillations, which can be calculated as solutions of the Cauchy problem for the system (3.22). Hamiltonians of economic origin are characterized by the absence of extrema. Their critical points are saddles, and all paths with bounded on a large interval dual variables are close to the stable separatrices of the saddles. A separatrix obviously does not depend on the initial and final conditions, but to find it, one needs to solve not the Cauchy problem, but a **boundary problem**.

3.5. Interaction of Capital Control. In the deterministic case, which is in question here, the principle of rational expectations leads to a model of *intertemporal economic equilibrium* [46, Intertemporal Equilibrium]. In such a model, each agent, based on her goals, opportunities, and forecasts, defines her supply/demand for products, resources, and financial instruments at the current and all future moments, for example, as a solution $y_a(t) = \hat{y}_a[\mathbf{r}(\cdot), \mathbf{p}(\cdot)](t)$ to a problem of the type (3.8)–(3.14) depending on

[12] For example, in problem (3.7) we will get condition $\xi_a(T) = 0$, which contradicts (3.11) and (3.6) and the requirement of absolute continuity of $\xi_a(\cdot)$, which in turn is necessary to perform integration by parts in (3.7).

[13] At first we state the condition of growth of linear form of assets with unspecified coefficients. The problem appears to be solvable only if this form is proportional to capital form (3.18) [25].

the observed and planned indicators of the economic state $\mathbf{r}(t)$, $\mathbf{p}(t)$. As to trajectories of $\mathbf{r}(t)$ and $\mathbf{p}(t)$, they are defined so that demand of one agent matches supply of others in the framework of the complete system of balances (if there are only two agents a and b in the model, as in Fig. 2.2.1, then $\hat{y}_a[\mathbf{r}(\cdot), \mathbf{p}(\cdot)](t) + \hat{y}_b[\mathbf{r}(\cdot), \mathbf{p}(\cdot)](t) = 0$). Intertemporal equilibrium models have long been known, but so far they have been used exclusively for the study of theoretical issues in stationary modes of rather abstract models of the economy. We took a chance to apply this approach to model the observed dynamics of the Russian economy, and, as will be shown below, after some theoretical findings and overcoming of serious difficulties in implementation, it led to success.

Perhaps the most important discovery of mathematical economics during all its lifetime is that the set of agents in such a model can be, in a sense, derived from the requirements of economic efficiency. Let us imagine that we started planning the perfect economy and for this purpose we have:

- identified final consumers in the economy and assigned them utility functions of type (3.4), but dependent not on the scalar cash flow but on the vector flow of consumed material goods;
- identified from the descriptions of all the agents all the technological limitations on production of material goods;
- written down all material balances;
- set a problem of maximizing the weighted sum of final consumer utility, given all technological and balance constraints.

Then it turns out [46, Welfare Theorem], that, by removing the balance constraints with Lagrange multipliers and using the saddle point theorem (3.7), we can transform the problem of centralized planning into a problem of finding the Nash equilibrium in noncooperative game of consumers, producers,[14] and merchants.[15]

Thus, we can say that the model of intertemporal equilibrium is obtained not by assembly of blocks, like simulation model, but instead by decomposition of integral system of balances. In the process of construction of a realistic model, this decomposition is distorted due to introduction of additional institutional constraints.

However, in this remarkable decomposition

(a) the problem that producers are solving is not of form (3.4), but is a problem of maximizing summarized by time profit, calculated in the dual variables to the balances;
(b) the total profit of all producers should be distributed among consumers according to the weighting factors of consumers utility, set in the original planning problem. But the decomposition does not define how the payments of the profits are distributed among moments of time and among producers.

We see the mentioned ambiguity as an advantage, rather than a shortcoming, of decomposition. It means that at an idealized level, there are many different ways of organizing the financial system, that ensure the optimal development regime of the economy. But if we want to build an applied model based on this theory, we have to resolve the ambiguity somehow.

The intertemporal equilibrium models, known to us from the world practice, including all models from the fundamental work [53] are based on the following solution, proposed by M. Sidrauski [52]: if all factors of production—raw materials, natural resources, labor, and capital[16]—are purchased (or rented) by Producer and technological constraints are homogeneous, then the maximum profit is zero. Therefore, the question of the distribution of income is removed, and the problem of optimal planning for producer

[14]The minimum number of agents is determined by the structure of technological limitations. After conversion of dual variables to balances into price-like variables, commercial banks and the Central Bank will be added to this set of agents [27]. If we consider the production of not only additive goods, but the public goods as well, then the state can be, to some extent, added as an agent.

[15]A Merchant mentioned in the list of Sec. 2.2 and described in Sec. 6.6, is not the same agent that emerges from decomposition of the planning problem.

[16]Buildings, structures, machinery and equipment [32]. Do not confuse it with own capital discussed in Sec. 3.3. Fixed capital owned by the company is its asset while own capital is a liability (liability to the owner).

turns to the problem of maximizing current income at each moment of time. But the owner of capital now has the problem of controlling this capital through investment. The owner of capital is the Household, and capital is simply identified with the real value of savings.

In our view, such a decision, elegant as it is, is not quite satisfactory. First, modern owners (shareholders) do not make decisions on specific project investment. Capital management takes place at a purely financial level and specific investment decisions are made by managers of firms (firms own fixed capital). Second, the identification of capital with savings excludes banking and in general the entire financial system, from the model. Its economic function is exactly to transform savings into investments. Therefore, in the models that use the Sidrauski scheme, there are no loans and deposits, and the dynamics of credit and the monetary system are reduced to dynamics of liquidity.

In [27] I. G. Pospelov proposed an alternative way of describing relations between the owner and the company in an ideal intertemporal equilibrium model. It was called the intertemporal equilibrium model with control of capital (IEMCC). In this construction, it is assumed that the owner o of the company a (we assume here for simplicity that the firm has a single owner) sets the task to the manager of the firm of maximizing the flow of dividends $u_a(t)$ in a given time ratio $\nu_a(t) > 0$,

$$u_a(t) = \bar{\theta}_a \upsilon_o(t), \quad \bar{\theta}_a \to \max . \tag{3.23}$$

This problem can be seen as a limiting case of the problem (3.4) provided

$$U(t,u) = \frac{(u/\upsilon_o(t))^{1-\eta}}{1-\eta}, \quad \eta \to \infty.$$

Solving the problem (3.23) under constraints (3.1)–(3.3), the firm determines the value of $\bar{\theta}_a$ and return rate (3.17) of its capital (3.18):

$$\upsilon_o(\cdot) \Rightarrow \bar{\theta}_a, \rho_a(\cdot). \tag{3.24}$$

It is assumed that the firm reports these data to its owner, and that allows her to plan the dynamics of the value $K_o(t) = \Omega_a(t)/\bar{\theta}_a$ by (3.19) and revenue from firm $Z_o(t) = u_a(t)$:

$$\frac{d}{dt} K_o(t) = \rho_a(t) K_o(t) - \upsilon_o(t), \quad K_o(t) \geq 0, \quad Z_o(t) = \bar{\theta}_a \upsilon_o(t). \tag{3.25}$$

The value of $K_o(t)$ is in some sense the owner's investment (share) in firm capital, and the value of $\bar{\theta}_a$ plays the role of the price of these securities. The first relation in (3.25) shows that investments grow with the firm's return rate and decrease while extracting profit.

After including relations (3.25) for asset $K_o(t)$ and flow $\upsilon_o(t)$ into constraints, and after taking into account the inflow of income $Z_o(t)$ in money balance, and after solving the problem (3.4), the owner will determine the optimal (for herself) program of $\upsilon_o(t)$:

$$\bar{\theta}_a,\ \rho_a(\cdot) \Rightarrow \upsilon_o(\cdot). \tag{3.26}$$

The fixed point of the superposition of maps (3.24) and (3.26) specifies a certain informational equilibrium, which we take as a description of interaction between the firm and its owner in IEMCC. In this interaction, the fixed capital belongs to the firm, which makes investment decisions, while the owner manages the firm's own capital relying solely on financial characteristics $\bar{\theta}_a$ and $\rho_a(\cdot)$. In this scheme in the framework of the decomposition of the planning problem in the perfect intertemporal equilibrium model there emerge money debts and savings, as well as banks. Some confusion is caused by the fact that the firm's price $\bar{\theta}_a$ remains constant, but it is necessary in a deterministic model. It becomes variable only in stochastic version of IEMCC [6].[17]

Interestingly, given homogeneous utility functions for final consumers, their problems of the form (3.4) are reduced to the form (3.23). It is only that the time proportion of their utility expenses is determined not by the owner, but by market conditions [7]. Thus, in the homogeneous IEMCC, descriptions of

[17] Recently N. P. Pilnik proposed a new scheme of decomposition, in which a mechanism of issuing and trading shares is introduced [24]. In this scheme, a firm solves an arbitrary nondegenerate problem of form (3.4), and yet the intertemporal equilibrium is effective (equivalent to central planning). Currently this scheme is still undergoing testing.

interests of agents are unified: we can assume that the producers and consumers seek to maximize their appropriately understood capitalization. However, in Sec. 6, which deals with specific descriptions of agents, we do not exploit this unification.

From a theoretical point of view we can consider the Household as the the owner of Producer and Bank. But we highlight the owner of capital as a special agent (see Sec. 2.2), consistent with Russian reality: common people practically do not invest in stocks (and therefore the stock market plays a minor role in the economy).

4. Model Products

4.1. Single-Product Model Insufficiency. An important feature of the model is the original method of calculation of aggregate products. To explain how it works, let us return to the scheme of allocation of material goods, described by the balances shown in Fig. 2.1.1. Let us imagine that we can trace the origin and utilization of the entire set $\mathfrak{G}$ of products (goods and services) that circulate in the economy for some time from initial $\tau = t_0$ to the current period $\tau = t$. Here we can distinguish between imported $g \in \mathfrak{I}$ and domestic products, and among the latter those intended for export $g \in \mathfrak{E}$, and for internal use $g \in \mathfrak{X}$, $\mathfrak{G} = \mathfrak{I} + \mathfrak{E} + \mathfrak{X}$. Then, summing up the material balances from Fig. 2.1.1 by all agents in the country and by all their external counterparts, as well as combining capital costs and stocks growth into a single value of investment, we obtain the following detailed material balances[18]:

$$X^g(t) = E^g(t), \quad g \in \mathfrak{E}; \tag{4.1}$$

$$X^g(t) = X^g_V(t) + X^g_C(t) + X^g_J(t), \quad g \in \mathfrak{X}; \tag{4.2}$$

$$I^g(t) = I^g_V(t) + I^g_C(t) + I^g_J(t), \quad g \in \mathfrak{I}. \tag{4.3}$$

Here $X^g(t)$, $I^g(t)$, and $E^g(t)$ are the volumes of, respectively, production, import, and export of product g, while variables indexed with V, C, and J are the volumes of use of this product as intermediate consumption, terminal consumption, and investment, respectively.

As mentioned above, the detailed balances are unobservable, and to obtain relations analogous to ones observed in the economy, these balances have to be aggregated with cost, that is, multiplied by product prices and summed by product groups. Unfortunately, the observed *purchasers' prices* are not suitable for this. The store buys from the producer and the consumer buys from the store at different prices, as the consumer pays not only for goods, but also for trade services. Therefore, the calculated statistics forms **basic prices**, excluding trade and transport margins of purchasers' prices and transferring them to the balance of services.[19] Obtained basic prices can already be considered the same for all agents; at the same time they reflect the cash flows as shown in Fig. 2.1.2.

Let us assume that we know the basic prices of all products $p_g(\tau)$, $g \in \mathfrak{G}$, at all periods of time. Let us also take a period t_0 for the base. Multiplying balances (4.1)–(4.3) by the prices of the base period $p_g(t_0)$ and summing by the groups $\mathfrak{I}$, $\mathfrak{E}$, and $\mathfrak{X}$, we obtain the aggregate balances

$$X^{\mathfrak{E}}(t) = E, \quad X^{\mathfrak{E}}(t) = \sum_{g\in\mathfrak{E}} p_g(t_0)X^g(t); \tag{4.4}$$

$$\begin{gathered} X = X^{\mathfrak{X}}(t) = X^{\mathfrak{X}}_V(t) + X^{\mathfrak{X}}_C(t) + X^{\mathfrak{X}}_J(t), \\ X^{\mathfrak{X}}_U(t) = \sum_{g\in\mathfrak{X}} p_g(t_0)X^g_U(t), \quad U = \varnothing, V, C, J; \end{gathered} \tag{4.5}$$

$$\begin{gathered} I = I^{\mathfrak{I}}_V(t) + I^{\mathfrak{I}}_C(t) + I^{\mathfrak{I}}_J(t), \\ I^{\mathfrak{I}}_U(t) = \sum_{g\in\mathfrak{I}} p_g(t_0)I^g_U(t), \quad U = \varnothing, V, C, J, \end{gathered} \tag{4.6}$$

[18] We neglect the value of re-export as it is very small in Russia.

[19] Taxes on products are also excluded, and to account for the difference between export and domestic prices, some artificial flows of export and import trade services are introduced.

where E and I are theoretical analogues of **real** (i.e., valued at base year prices) volumes of current **export** and **import**.

Now summing up balances (4.4)–(4.6) and introducing the notation for

- real value of **terminal consumption** (private and public)

$$C(t) = X_C^{\mathfrak{X}}(t) + I_C^{\mathfrak{I}}(t),$$

- real value of **gross fixed capital formation**, i.e., investment in stocks and fixed capital

$$J(t) = X_J^{\mathfrak{X}}(t) + I_J^{\mathfrak{I}}(t),$$

- **real GDP**, i.e., value at basic prices of newly produced products

$$Y(t) = X^{\mathfrak{E}}(t) + X^{\mathfrak{X}}(t) - X_V^{\mathfrak{X}} - I_V^{\mathfrak{I}}(t),$$

we get equality, that serves as a theoretical analogue of the national accounts by consumption (NAC) in real terms [32]:

$$Y(t) = J(t) + C(t) + E(t) - I(t). \tag{4.7}$$

The main task of macroeconomic modelling is analysis and forecasting of the individual components of this balance. Within the framework of a single product, where most macroeconomic models reside, NAC (4.7) is interpreted as a balance of parts of the single product between different agents, that is, actually as a condition of market equilibrium of this product. But in a competitive market the single product has the same price for all customers and sellers. Let us see how this idea relates to statistics.

The problem is that prices of individual products are equally unobservable as their flows. In fact, only cash payments are fully and systematically observed in the economy in the form

$$\tilde{C}(t) = \sum_{g \in \mathfrak{X}} p_g(t) X_C^g(t) + \sum_{g \in \mathfrak{I}} p_g(t) I_C^g(t).$$

As a result we can first get NAC at current prices $\tilde{Y}(t) = \tilde{J}(t) + \tilde{C}(t) + \tilde{E}(t) - \tilde{I}(t)$. Then, each component is divided into two factors: **base deflator** showing the mean growth rate of prices for products of this group, and **real volume**, evaluating growth in constant prices. As a result, NAC in current prices has the form

$$p_Y(t)Y(t) = p_J(t)J(t) + p_C(t)C(t) + p_E(t)E(t) - p_I(t)I(t). \tag{4.8}$$

At the same time relation (4.7) holds for the real volumes.[20]

Deflators $p_Y(t)$, $p_J(t)$, $p_C(t)$, $p_E(t)$, and $p_I(t)$ are defined as **price indices**, measured by samples (commodity baskets) of the goods [11], or using trade statistics [16]. Despite the variety of techniques of calculation of indices, they in practice produce very similar results, and the resulting time series of macroeconomic data [36] show nontrivial internal regularities (see, for example, Sec. 4.3), that allow us, from our point of view, to consider this statistic quite reliable and objective.[21]

It is clear that the relative accuracy of a single-product model in reproducing the dynamics of the NAC (4.7) components cannot exceed the relative difference between deflators, because a single-product model implies equal deflators. Figure 4.1.1 shows that from about 2004–2005 deflators in Russian economy began to diverge. This caused us to abandon the single-product model and develop a disaggregation method for macroeconomic balance, based on the difference in deflators.

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[20] In statistics, the right-hand side of both balances (4.7) and (4.8) contains small statistical discrepancies, since the left- and right-hand sides of the balances are measured independently. In processing the statistics, we subtract these divergences from GDP.

[21] In contrast with sectoral and regional statistics, that show a lot of quirks and contradictions. This paradox—statistics are collected from the bottom up, but the upper levels appear more reliable—from our point of view confirms the idea stated above that material balances nowadays lose their meaning. Upper level statistics is trustworthy since it relies more on financial indicators.

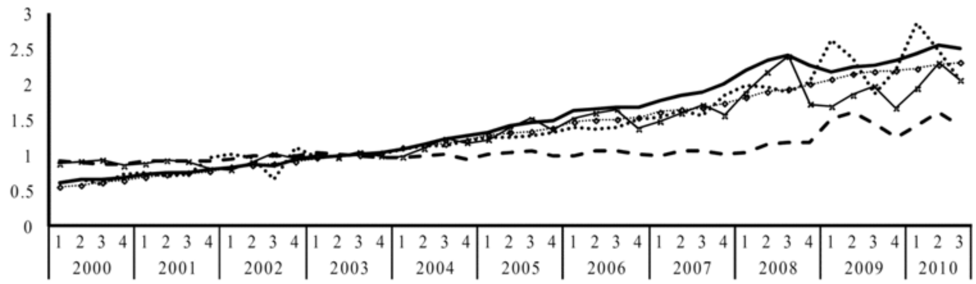

Fig. 4.1.1. GDP deflator (p_Y, without statistical discrepancy), terminal consumption deflator (p_C), gross fixed capital formation deflator (p_J), export deflator (p_E), import deflator (p_I).

4.2. Nonlinear Disaggregation of NAC. So, we no longer consider relation (4.7) to be the model balance of additive good. Instead, we assume that the components of NAC are nonlinear homogeneous aggregates of three additive goods of lower level: observed exported E and imported I product, and also unobserved domestic product X. However, before writing these convolutions, we make one further simplification.

We assume that the use of imported products (building materials, equipment, parts, licenses and patents on the technology, etc.) as raw materials and components is dictated not so much by technological limitations as by customer requirements. Based on this, we include intermediate components into final ones[22] and replace the balances (4.5) and (4.6) with model balances

$$X(t) = J_X(t) + C_X(t), \quad I(t) = J_I(t) + C_I(t), \tag{4.9}$$

where

- J_I—Imports associated with investment, such as import of equipment for the plant (final product) and imported materials for its construction (intermediate);
- C_I—Imports associated with consumption such as foreign car (final product) and imported picture tube (intermediate) for a TV-set assembly;
- J_X—Domestic production, associated with investment, such as construction services (final services) and electricity expended in construction (intermediate);
- C_X—Domestic production, associated with consumption such as trading services for sales of foreign cars (final services) and electricity expended in the provision of services (intermediate).

Further, we assume that export volume $E(t)$ and import volume $I(t)$ in (4.7) show full production and inflow/outflow of model products E, I; observed consumption (4.7) is expressed as

$$C(t) = g\big(C_I(t), C_X(t)\big), \tag{4.10}$$

and observed investment as

$$J(t) = h\big(J_I(t), J_X(t)\big). \tag{4.11}$$

Finally, we assume that the real GDP can be implemented in various combinations of domestic and export products, so

$$Y(t) = f\big(X(t), E(t)\big). \tag{4.12}$$

Functions $g(\cdot,\cdot)$, $h(\cdot,\cdot)$, and $f(\cdot,\cdot)$ are assumed to be, at least, monotonous and *linear homogeneous*. The last requirement is natural for aggregating expressions because it ensures their independence from the arbitrary choice of units. As shown in [10], homogeneity ensures addition of financial flows in the nonlinear convolution of real flows.

[22] Test calculations with data on the intermediate products that recently appeared in [36] have shown that their inclusion does not improve the model.

The assumption of partial substitution of domestic and imported products has long been used in models of foreign trade under the name of "Armington assumption" [43]. Description of production and consumption in the very detailed (hundreds of products and resources) Australian model [45] is based on a hierarchy of nonlinear convolutions. The authors, however, determine the parameters of convolution using a single point and complain about their instability. In CGE model [44] three-stage budgeting of households was used. In the first phase, expenditures are shared between the types of products; in the second phase, expenditures for each type of product are divided between domestic and imported representatives of each type; and in the third phase, import expenditures for each type of product are distributed among producing countries.

4.3. Rationality of Behavior of Economic Agents. Since time series of $X(t)$, $J_X(t)$, $C_X(t)$, $C_I(T)$, $J_I(t)$ are not available in statistics, the relations (4.9)–(4.12) are insufficient to identify the functions $g(\cdot,\cdot)$, $h(\cdot,\cdot)$, $f(\cdot,\cdot)$ in (4.10)–(4.12). However, we have unused information about deflators (see Fig. 4.1.1), from which, in fact, the discussion on disaggregation started. This information may be exploited if we interpret the aggregation functions $g(\cdot,)$, $h(\cdot,\cdot)$, $f(\cdot,\cdot)$ as **utility functions** or **costs**. For this analysis, we have to introduce one more deflator, missing in statistics—**domestic product deflator** $p_X(t)$.

Homothetic to one another level lines of $g(\cdot,\cdot)$ function can be seen as lines of indifference, that show the possibility of substitution between domestic and imported product to the aggregated final consumer[23] and the function $g(\cdot,\cdot)$ itself—as a purchase utility function.[24] Having bought products $C_I(t)$ and $C_X(t)$ in period t, the consumer incurred cash expenses $\Phi(t) = p_I(t)C_I(t) + p_X(t)C_X(t)$. Assuming (4.10) we get consequently

$$p_C(t)g\big(C_I(t), C_X(t)\big) = p_I(t)C_I(t) + p_X(t)C_X(t). \tag{4.13}$$

Why was a set of products $C_I(t)$, $C_X(t)$ bought, and not another one? The simplest answer is that given a different set of products and invariable prices, the relation (4.10) is incompatible with the budget constraint

$$p_C(t)\, g(C_I, C_X) \le p_I(t)\, C_I + p_X(t)\, C_X \quad \text{for all} \quad C_I, C_X \ge 0. \tag{4.14}$$

It follows from relations (4.13) and (4.14), that values of $C_I(t)$ and $C_X(t)$ provide the maximum to expression

$$p_C(t)\, g(C_I, C_X) - p_I(t)\, C_I - p_X(t)\, C_X \to \max_{C_I, C_X \ge 0}. \tag{4.15}$$

For the problem (4.15) to be correct, we make the usual assumption for the utility function that it is *smooth and concave*. Then, given that we are interested only in internal maximum points[25] we obtain necessary and sufficient conditions for (4.15)[26]:

$$\partial_1 g\big(C_I(t), C_X(t)\big) = \frac{p_I(t)}{p_C(t)}, \quad \partial_2 g\big(C_I(t), C_X(t)\big) = \frac{p_X(t)}{p_C(t)}. \tag{4.16}$$

However, since the function $g(\cdot,\cdot)$ is linearly homogeneous, the maximization problem (4.15) is degenerate. It has a nontrivial solution for not all relative prices, and when the solution exists, it is not unique. In this case, at the maximum point it is necessary that the equality (4.13) holds. Algebraically it is manifested in that the partial derivatives of $g(\cdot,\cdot)$ are functions of ratios of arguments. Therefore, the equations (4.16) define not the values of $C_I(t)$ and $C_X(t)$, but the ratio $C_I(t)/C_X(t)$ and the price $p_X(t)$. For reasons of symmetry it is more convenient to use the ratio of equalities (4.16) and condition (4.13):

$$\begin{gathered} \frac{\partial_1 g(C_I(t), C_X(t))}{\partial_2 g(C_I(t), C_X(t))} = \frac{p_I(t)}{p_X(t)}, \\ p_C(t)\, g\big(C_I(t), C_X(t)\big) = p_I(t)\, C_I(t) + p_X(t)\, C_X(t). \end{gathered} \tag{4.17}$$

[23]Substantially final consumer combines household and the state (private and public consumption). The natural attempt to divide them by introducing two different aggregation functions for public and private consumption, to our surprise, was unsuccessful.

[24]The treatment of homogeneous utility functions as aggregate goods is considered in [16, 20].

[25]Purchases of solely domestic or solely imported product do not comply with the statistics.

[26]The symbol ∂_i denotes the partial derivative with respect to the ith argument.

Considering the concave smooth function $h(\cdot,\cdot)$ in (4.11) as a utility function of an aggregate investor and using similar considerations, we get the relations

$$\begin{gathered} \frac{\partial_1 h(J_I(t), J_X(t))}{\partial_2 h(J_I(t), J_X(t))} = \frac{p_I(t)}{p_X(t)}, \\ p_J(t)\, h\big(J_I(t), J_X(t)\big) = p_I(t) J_I(t) + p_X(t) J_X(t). \end{gathered} \tag{4.18}$$

For the function $f(\cdot,\cdot)$, arguments change somewhat. The value of $V(t) = p_X(t)X(t) + p_E(t)E(t)$ is the producer's revenue. In statistics it is expressed as $p_Y(t)Y(t) = p_Y(t)\, f\big(X(t), E(t)\big)$, and it can be interpreted as the cost of acquisition of the production factors required for the production of $Y(t)$. Then a rationally acting aggregate producer must choose a combination of volumes $X(t)$ and $E(t)$, that maximizes current profit on sales of goods manufactured:

$$p_X(t)\, X + p_E(t)\, E - p_Y(t)\, f(X, E) \to \max_{X,E}. \tag{4.19}$$

This problem is well posed, if $f(\cdot,\cdot)$ is **convex**. In view of the homogeneity of $f(\cdot,\cdot)$ the maximum profit is zero, and we arrive again at conditions similar to (4.17) and (4.18):

$$\begin{gathered} \frac{\partial_1 f(X(t), E(t))}{\partial_2 f(X(t), E(t))} = \frac{p_X(t)}{p_E(t)}, \\ p_Y(t) f\big(X(t), E(t)\big) = p_X(t)X(t) + p_E(t)E(t). \end{gathered} \tag{4.20}$$

At last, we obtained **eleven** relations (4.9)–(4.12), (4.17), (4.18), and (4.20) on **six** time series absent in statistics: $X(t)$, $C_X(t)$, $J_X(t)$, $C_I(t)$, $J_I(t)$, and $p_X(t)$. Note, that NAC (4.7) and (4.8) hold, because they contain only statistical time series. If we can find the time-independent functions $g(\cdot,\cdot)$, $h(\cdot,\cdot)$, and $f(\cdot,\cdot)$, under which this overdetermined system of conditions holds with good accuracy for each time, then we get solid confirmation of the possibility to correctly disaggregate NAC into three model products. Note that at the same time, we get confirmation of the general provision on the possibility to describe the behavior of macroagents as rational.

It should also be noted that arguments about detailed balances given in Sec. 4.1 can be regarded as simply suggestive considerations and the relations (4.9)–(4.12) and (4.17)–(4.20) as a purely phenomenological scheme, almost uniquely determined by the requirement to describe the components of the macroeconomic balance as a nonlinear convolution of the minimum number of additive goods under the conditions of rationality.

4.4. Parametrization and Identification of Disaggregating Functions. We seek functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$, and $h(\cdot,\cdot)$ in the form

$$\begin{aligned} f(X,E) &= A_f(\sigma_f X^{e_f} + (1-\sigma_f)\, E^{e_f})^{1/e_f}, \\ h(J_I, J_C) &= A_h(\sigma_h J^{e_h} + (1-\sigma_h)\, J^{e_h})^{1/e_h}, \\ g(C_I, C_X) &= A_g(\sigma_g C_I^{e_g} + (1-\sigma_g)\, C_X^{e_g})^{1/e_g}. \end{aligned} \tag{4.21}$$

Functions of this form (constant elasticity of substitution—CES) are widely used in mathematical economics. Under appropriate parameters they give expression for various averages, as well as the popular Cobb–Douglas function. Obviously these functions are linearly homogeneous. When the exponent is greater than one, the CES function is concave, and when the exponent is less than one, it is convex. So, different requirements on the direction of convexity of functions $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ on one hand, and $f(\cdot,\cdot)$ on the other, may be taken into account by the choice of parameter values.

Now we get the problem of identification, which can be informally stated as follows: choose the coefficients A_f, σ_f, e_f, A_h, σ_h, e_h, A_g, σ_g, and e_g in (4.21) so that given known statistical time series

$$Y(t),\ J(t),\ E(t),\ I(t),\ p_Y(t),\ p_J(t),\ p_C(t),\ p_E(t),\ p_I(t)$$

the overdetermined system of relations (4.9)–(4.12) and (4.17)–(4.20) would hold as accurately as possible. For a correct formulation of this problem it is necessary to determine the meaning of "as accurately as possible."

We assume that the financial balances (the second equations in (4.17)–(4.20)) and material balances (4.9) should hold strictly. In conditions of aggregating (4.10)–(4.12) and conditions of rational behavior (the first equations in (4.17)–(4.20)) we allow deviation. Since the absolute values on the left-hand sides of relations (4.10)–(4.12) grow in time with the pace of economic growth, while the ratios of prices on the right-hand side (4.17)–(4.20) remain nearly constant in time, we will evaluate deviations in aggregating conditions on a logarithmic scale, and the deviations in rationality conditions — on a natural scale. Thus, we replace Eqs. (4.10)–(4.12) with

$$\begin{aligned}
\ln Y(t) &= \ln A_f + \frac{1}{e_f}\,\ln(\sigma_f X^{e_f} + (1-\sigma_f)\,E^{e_f}) + \varepsilon_1(t),\\
\ln J(t) &= \ln A_h + \frac{1}{e_h}\,\ln(\sigma_h J_I^{e_h} + (1-\sigma_h)\,J_C^{e_h}) + \varepsilon_2(t),\\
\ln C(t) &= \ln A_g + \frac{1}{e_g}\,\ln(\sigma_g C_I^{e_g} + (1-\sigma_g)\,C_X^{e_g}) + \varepsilon_3(t),
\end{aligned} \tag{4.22}$$

and we replace the first equations in (4.17)–(4.20) with

$$\begin{aligned}
\frac{\sigma_f}{1-\sigma_f}\left(\frac{X(t)}{E(t)}\right)^{e_f-1} - \frac{p_X(t)}{p_E(t)} &= \varepsilon_4(t),\\
\frac{\sigma_h}{1-\sigma_h}\left(\frac{J_I(t)}{J_X(t)}\right)^{e_h-1} - \frac{p_I(t)}{p_X(t)} &= \varepsilon_5(t),\\
\frac{\sigma_g}{1-\sigma_g}\left(\frac{C_I(t)}{C_X(t)}\right)^{e_g-1} - \frac{p_I(t)}{p_X(t)} &= \varepsilon_6(t).
\end{aligned} \tag{4.23}$$

Further, we will minimize the mean square error (residual)

$$\sum_{t=1}^{T}\sum_{i=1}^{6}\varepsilon_i(t)^2 \to \min \tag{4.24}$$

with respect to $X(t)$, $C_X(t)$, $J_X(t)$, $C_I(t)$, $J_I(t)$, $p_X(t)$, $\sigma_f, \sigma_h, \sigma_g, A_f, A_h, A_g > 0$, e_f, e_h, e_g and $\varepsilon_i(t)$ for $i = 1, \dots, 6$, $t = 1, \dots, T$, where T is the length of the statistical time series, under constraints

$$\begin{aligned}
X(t) = J_X(t) + C_X(t), \quad I(t) &= J_I(t) + C_I(t),\\
p_C(t)g\big(C_I(t), C_X(t)\big) &= p_I(t)C_I(t) + p_X(t)C_X(t),\\
p_J(t)h\big(J_I(t), J_X(t)\big) &= p_I(t)J_I(t) + p_X(t)J_X(t),\\
p_Y(t)f\big(X(t), E(t)\big) &= p_X(t)X(t) + p_E(t)E(t).
\end{aligned} \tag{4.25}$$

Equality constraints (4.25) allow us to rule out some of the variables to reduce the problem (4.24) with $(9 + 6T)$ variables to a problem with nine unknown parameters of production functions (4.21). Nevertheless the problem is quite complicated, so custom algorithms were used to approach its solution. They were run on a supercomputer MVS100K at the Joint Supercomputer Center (JSCC) of RAS.

Parameters were identified using quarterly NAC data without seasonal adjustment in the current market prices (nominal values) and in average market prices of year 2003 (real values) for the period from I-2000, to IV-2010 [36] — 44 observation points overall. The optimal parameters in (4.21) give the following form of aggregate functions:

$$f = 1.88\,(0.68X^{1.05} + 0.32E^{1.05})^{\frac{1}{1.05}},$$

$$h = 1.75(0.22J_I^{0.005} + 0.78J_C^{0.005})^{\frac{1}{0.005}}, \quad g = 1.85(0.28C_I^{0.003} + 0.72C_X^{0.003})^{\frac{1}{0.003}}.$$

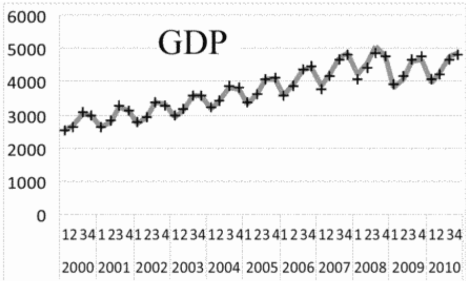

Fig. 4.4.1. GDP (4.21).

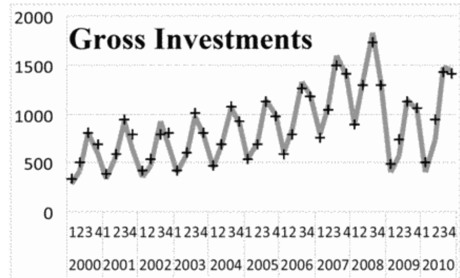

Fig. 4.4.2. Fixed capital formation (4.21).

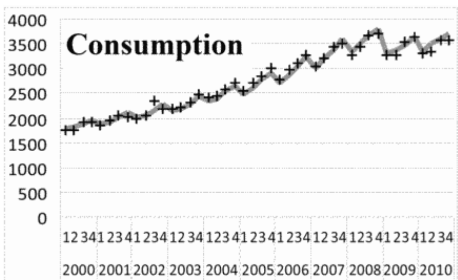

Fig. 4.4.3. Terminal consumption (4.21).

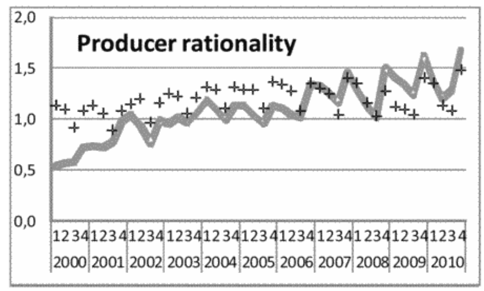

Fig. 4.4.4. Rationality of producer (4.23).

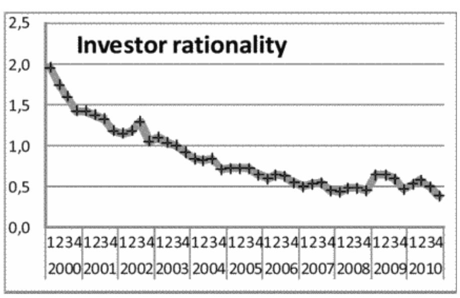

Fig. 4.4.5. Rationality of investor (4.23).

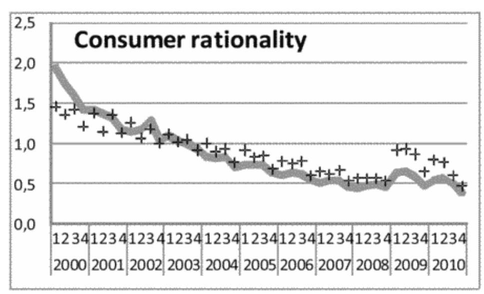

Fig. 4.4.6. Rationality of consumer (4.23).

The function f appeared to be convex, and the functions g and h appeared concave as they should be in this context, although it was not required by the equations formally. The achieved accuracy of the conditions of aggregation and rationality is illustrated in Fig. 4.4.1–4.4.6.

We see that the relations (4.22) hold almost exactly, including the period of financial crisis. This, among other things, shows that the particular way (4.24) for combining errors of individual equations into error functional, most likely, did not play a special role. Rationality conditions (4.23) hold less accurately, but in this case nevertheless, both sides of these conditions have a common nontrivial trend and a common phase of seasonal oscillations. Differences relate mainly to amplitude. Calculations with more realistic models of general equilibrium suggest that this discrepancy may be due to the fact that conditions of rationality do not take the effect of distorting taxes (first of all, import and export duties) into account.

From an empirical point of view, leaving aside the unobservable variables we introduced, the identified relations **implicitly** set four *nonlinear* dependencies for ten observable series: real GDP $Y(t)$, gross investment $J(t)$, consumption $C(t)$, export $E(t)$, import $I(t)$, and their deflators $p_Y(t)$, $p_J(t)$, $p_C(t)$, $p_E(t)$, and $p_I(t)$. These relations hold rather accurately on unsmoothed statistical data over the period of 44 quarters, including the crisis. It is a strong argument against doubts often expressed on the quality of the official statistical data. For these dependencies to be revealed, the statistics have to be, at least, integral and organized consistently.

In the model of the Russian economy the above relations form a rather artificially separated block "Trader" (see Sec. 6.6) to simplify the description of the Producer's behavior.

5. ECOMOD—Economic Modelling Assistance System

5.1. The Need for an Assistance System. An intertemporal equilibrium model is a combination of optimality conditions of all macroagents (Sec. 3.1) with scenario descriptions of individual agents and their interactions (IA blocks in Fig. 2.2.1). The latter contain, first of all, balances of financial and material flows between agents, and information exchange (Sec. 3.5).

A realistic model of this type may be reduced to a boundary value problem, which contains not only finite and differential equations, but also complementary slackness conditions (see 3.10). To deal with it, we have to use rather intricate algorithms [7]. At the same time, the number of adjustment parameters ranges between two and three dozen, which is a moderate number compared with hundreds in econometric, simulation, or CGE models. Nevertheless, when an intertemporal equilibrium model performs poorly, it demands a change in the form of the relations and the recomputation of everything from scratch. It is almost impossible to do this work on paper. One of the most efficient ways to work with a model is to use the ECOMOD assistance system [7], written in computer algebra system Maple [49]. This system is, in turn, based on the formalization of the notion of *the model's canonical form* (see Sec. 2.2) as a structured system of relations.

Structural and classification characteristics of the canonical form (types of variables and relations) are specified in ECOMOD either by a special notation of the variables or by special records in the headings of the groups of relations. The relations are written in the usual mathematical notation and may contain long text comments without any special signs. That is why the model in the ECOMOD system looks almost like an article or report, containing detailed descriptions of the model's blocks (see Sec. 6).

ECOMOD facilitates all stages of work with the model — from writing the relations to computations and analysis of results.

5.2. ECOMOD Functions. ECOMOD contains six levels of checking of the accuracy of the model in canonical form. Levels 1–5 verify in a strictly formalized way if the axioms of the canonical form hold.

(1) **Syntactic check**. Compliance with the usual rules of mathematical notation: bracket matching, presence of operands of binary operations and relations, constant number of functions' arguments, etc.

(2) **Balance check.** A balance equation must have the following form:

$$\text{rate of change in stocks or } 0 = \text{algebraic sum of flows} \tag{5.1}$$

(see Fig. 2.1.1, 2.1.3).

Flows describe how products appear or disappear in the economic system due to production or consumption (analogous to *sources* and *sinks*), and how products and instruments *transfer* from one agent to another. Since a balance equation describes a flow of additive quantity, and the model contains the complete system of its balances, the flow which describes transfer must emerge in another balance equation with the opposite sign.

The subsystem of balance equations, connected by transfer flows, describes circulation of some *asset*. Graphic representation of this subsystem is a typical flow scheme, which traditionally illustrates economic models. ECOMOD creates these flow schemes automatically based on balance equations.

Despite its simplicity, balance check provides amazing efficiency. Indeed, it is almost impossible to write by hand the balances of, say, five assets and three agents. And the mistakes are often revealed, unfortunately, only at the final computation stage.

(3) **Dimension check** is based on the simple idea that summation or comparison of values of different dimensions is meaningless, because the result depends on the chosen units of measurement (grams or tons,

roubles or kopecks). From the more general mathematical point of view, correct dimension of the system of relations means its invariance under certain group of similarity transformations [13].

For reasons not well understood, dimension check is rarely used in applied systems, and even if it is, it is reduced to the requirement of specification of all units of measurement in terms of some fixed system of basic units. We do not follow this procedure when we construct the canonical form of the model. First, the specification of a dimension for every value is cumbersome. Second, in economics, unlike physics, there are no independent units of measurement, except for time.

In the framework of the canonical form, it is natural to set independent dimensions for those assets, whose motion is described by balance equations, because these balance equations form the most reliable backbone of the model's relations. Within the canonical form we require that the *name* and *dimension* be defined for every asset. Different assets may have the same dimension: for example, loans and cash. The dimensions of different stocks of the same asset must coincide. The dimension together with the name of the asset is set in the heading of each balance equation.

The dimensions of assets and *time* form the core system of model dimensions. Flows have the dimension of assets divided by time. Dimensions of other variables and coefficients are derived from the base ones by simple rules using the model relations. If there is a mismatch when quantities of different dimensions are added or compared, the ECOMOD system signals about it.

Dimension mismatch usually means that some factor is missing in some relation. In practice, dimension check is a powerful and efficient debugging modelling tool, and we strongly recommend its use in applied systems. Used together, balance check, which, roughly speaking, finds missing terms, and dimension check, which finds missing factors, allows the elimination of the overwhelming majority of mistakes occurring in the equations.

(4) **Information communications check.** The canonic form of the model considers an agent as a decision-maker. An agent makes its decisions based on some limited information. For example, an agent normally does not know the plans of another agent. If we want to describe the case when she knows this information, we have to specify how this information is transferred and introduce a special informational variable. These informational channels are checked in ECOMOD by special agreement about the indexation of variables.

(5) **Semantic check.** Performing the above checks, ECOMOD simultaneously forms the database about parameters, variables, functions, balances, assets, and blocks of the model. This database also contains **names of variables**, that describe their meaning, and the **original form and class of relations**. Based on the collected information, the ECOMOD system builds a block diagram of the model, which represents the agents, their interactions, and flows of assets.

A semantic check is based not on the formal structure of the model, but on substantive interpretation of its variables and blocks.

(6) **Support of analytical transformation and analysis of the model.** From the beginning, it was a peculiarity of our way in economic modelling that we do not just calculate some model, but first try to study it through analytical transformation and consider parts of the model and its special cases. Based on preliminary examination, the model is often modified even before calculations.

Due to the absence of a standard system of basic equations, the stage of analytical study of the model in a traditional sense is fraught with technical difficulties. It is nearly impossible to handle a system of more than ten nonlinear equations correctly in a reasonable time without a computer. It is even harder to repeat the calculations after modifying the model. The researcher is tempted to change the model, which may violate the harmony of initial hypotheses.

Modern systems of computer algebra allow us to overcome these difficulties by automatically repeating long sequences of calculations after modification of the system. And the detailed account of relations and variables collected earlier is especially useful here, because it allows us to find the origin of the transformed relation.

Of course, a computer system cannot build a model. But it can store a lot of drafts and old models. The system can check if the old blocks fit the new model. Correspondence of the same names to various objects in different models partially shapes the categorical structure of the systems of models [39].

5.3. The Technology of Model Creation. The combination of ECOMOD checking and storage functions, the capabilities of Maple, and the unification of agents' descriptions based on the new formalization of capital opened the way to the new technology of economic modelling. This technology integrates and automates the following stages of model development.

(1) Blocks of the model, which describe behavior of agents, are written in standard mathematical notation in Maple environment with balance equations separated from institutional and technological constraints. After that, the equations are simplified **automatically** and reduced to a "human-friendly form," while the semantics of the original form remains unchanged. If the agent's behavior is described by the unified problem of maximizing the capitalization, the **system automatically writes down the sufficient optimality conditions** (Sec. 3.2). After that the conditions may be simplified with special procedures of elimination of the variables.

(2) Blocks, which describe the agents, are combined into one system, and the **automatic** redesignation procedures prevent contamination of names.

(3) Descriptions of agents' behavior are complemented by descriptions of their interactions, where their plans are harmonized by appropriate values of informational variables. Often, but not always, it is the balance of demand and supply established by equilibrium prices.

(4) Check of the system of balances, and derivation of the first integrals of the subsystem of financial balances. Both are carried out automatically.

(5) Simplification and study with analytical methods. From our experience, with special functions of displaying and excluding the variables, it is possible to work in Maple with nonregular systems with 100–150 equations and inequalities. It is important that the system remembers the original form of the relations, and the origin of the resulting relations can be easily established. It is also important that in most cases, calculations may be repeated after modification of the model.

(6) Calculation of dimension of variables allows us to automatize the search of self-similar solutions [7], which are used as the first approximation in calculations.

(7) **The calculation procedures use the expressions in mathematical notation without translating into programming language**. The latter process is not only extremely time-consuming, but also may lead to many errors that are difficult to control. Calculation of the solution is based on a special algorithm and is accompanied by the identification of some parameters. There is also a procedure for implementing the solution procedure on a supercomputer.

(8) The system allows conducting numerical experiments with the model. A representation of the model in the ECOMOD system and graphs of solutions are almost ready for publication.

6. The Blocks of the Model of the Russian Economy

6.1. The System of Notation. The following description of the model is, for the most part, taken from its implementation in ECOMOD system. According to the rules of canonical form, relations of the model are divided into blocks EA, describing the behavior of agents, and blocks IA, describing their interactions (Fig. 2.2.1). Each block has a short name, **which is used as the index of the variables in this block**. As was already mentioned, there are nine EA blocks in the model: *Producer* **J**; *Bank* **B**; *Household* **H**; *Owner* **C**; *Trader* **Q**; *State* **G**; *Central Bank* **CB**; *Importer* **IM**, and *Exporter* **EX**.

These agents are connected with seventeen IA: market of consumer product **c**; market of investment product **j**; market of manufactured product **y**; market of domestic product **u**; market of imported product **i**; market of exported product **e**; market of loans **l**; market of deposits **s**; loans of banks to Central Bank (CB) **cb**; public account (treasury) within CB **g**; labor market **r**; payment of taxes and receipt of subsidies **x**; producer's capital management **up**; bank's capital management **ub**; trader's management **uq**; currency market **w**; current account payments **n**.

All variables in the model have a *dimension*, base dimensions are **money** (billion roubles), **product** (billion roubles of 2003rd year), **time** (quarter), **labor** (million people), and **currency** (billion dollars).

Basic dimensions are defined in special balance equations of the form (5.1). In order to facilitate control of the balances, these equations must contain only the names of the flows on the right hand side, not their expressions through other variables. This requirement leads to the appearance of some "redundant" variables in the original record of the block. However, they are excluded automatically, and the simplified form of the expressions is provided in the end of the block.

In the description of the EA block the planned variables of the agent under consideration are not indexed, and all nonindexed variables are considered as planned when one varies the functional. The index is assigned to the planned variables automatically before assembling the model. The subsequent description complies with this convenient ECOMOD rule.

For readability, after assembling the model, the most important variables get nonindexed names (ECOMOD checks correctness of renaming). Exogenous variables, i.e., the ones that are not defined in the model neither by agents nor in interactions, but from the outside (external prices, rates of taxes), obtain names ending with the suffix "_*s*."

We use special notation for the complementary slackness conditions (3.10): a relation of the form $[a]\,[A]$ without the sign of equality or inequality means the system of relations $a \cdot A = 0$, $a \geq 0$, $A \geq 0$.

For technical reasons, inequalities in ECOMOD are written in braces. Just like every other system of computer algebra, Maple works better with integrals than with sums, so we formulate the model in continuous time and proceed to traditional discrete description with the step (tick) equal to the tick of statistical data only just before calculation.

The reference of the form (6.1)–(6.9) applies to all formulas between (6.1) and (6.9) in a separate line, whether they are numbered or not.

6.2. Macroagent Producer J.

6.2.1. Restrictions. Producer, which represents the aggregate of commercial nonfinancial organizations in the model,

- hires labor (IA **r**);
- increases fixed capital by acquisitions of investment product (IA **j**);
- produces all GDP using labor and capital and sells this output (IA **y**);
- gets interest-bearing loans from the Bank (IA **l**) and keeps in it the idle funds on a noninterest-bearing current account (IA **n**);
- pays taxes (IA **x**);
- pays dividends to Owner (IA **up**).

Production of real GDP is described with the very simple and, at the same time, unusual production function

$$Y(t) = A \cdot M(t) + B \cdot R(t) \cdot e^{b(t-t_0)}, \tag{6.1}$$

where $R(t)$ is the employed economically active population [36], and $Y(t)$ is the real GDP. Parameter b may be considered as a rate of exogenous technical progress (increase in labor productivity). $M(t)$, which we do not associate with statistics[27], represents the producer's effective capital stock, which changes according to the equation

$$\frac{dM(t)}{dt} = J(t) - \kappa M(t), \tag{6.2}$$

where $J(t)$ is real gross investment.

We use relations (6.1) and (6.2) just because they describe unsmoothed dynamics of GDP much better than the usual formulas. Figure 6.2.1 shows the comparison of GDP and the estimation based on formulas (6.1) and (6.2) with discretization of (6.2) with a tick of 1 quarter with $A = 4.48$, $B = 32.78$,

[27]Unlike other countries, attempts to use fixed capital to describe the dynamics of production in Russia failed systematically [9].

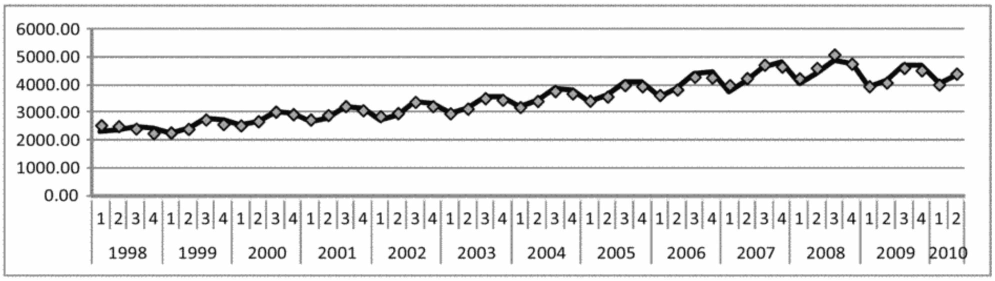

Fig. 6.2.1. Identification of production function. Black solid line is statistical data (left-hand side (6.1)), diamonds are calculated values (right-hand side of (6.1) provided (6.2)).

$b = 0.0094$, and $\kappa = 4.98$. The meaning of these relations, including the unusually large value of disposal rate $\kappa = 4.98$, is discussed in [7].

Public investments by State play an important role in Russia. We assume that the State purchases investment product $J_j(t)$ using budget funds and places it under control of Producer. Producer combines it with its own investments. Hence Producer's payments for the investment product are

$$pJ(t) = p_j(t)\big(J(t) - J_j(t)\big), \tag{6.3}$$

where $p_j(t)$ is the price of the investment product in IA **j**.

Labor $R(t)$, hired by Producer, is paid at the wage rate $s_r(t)$, defined in the block IA **r**. Thus, the total wage payments are equal to

$$sR(t) = s_r(t)\,R(t).$$

Producer sells the output in the market **y** at the equilibrium price $p_y(t)$, defined in the same block IA. Revenue from sales of GDP is

$$pY(t) = p_y(t)Y(t).$$

Producer gets a loan $L(t)$ from Bank with average interest rate $r_l(t)$ for the period around $\beta_l(t)^{-1}$. It appears to be essential to take into account how the average duration of loans depends on time, and we assume this dependence to be exogenous [4]. It is also important that a fraction of loans is never returned and is being written off. In order to account for this, the process of lending is described with relations

$$\begin{gathered} \frac{d}{dt}\,L(t) = K(t) - HL(t) - ML(t), \quad K(t) \geq 0, \\ ML(t) = \sigma L(t), \quad HL(t) = \beta_l(t)L(t), \quad WdL(t) = K(t) - HL(t), \quad rL(t) = r_l(t)L(t), \end{gathered} \tag{6.4}$$

where $K(t)$ is newly issued loans, $HL(t)$ is settlement of loans, $ML(t)$ is debt write-offs, which we assume to be proportional to the volume of loans, $WdL(t)$ is the balance of loans and loan settlements, and $rL(t)$ is interest payments.

Producer uses the bank current accounts $N(t)$ for financial transactions. We assume that macroagent constantly needs some positive $N(t)$ to buy the investment product $J(t)$, pay the taxes $Tax(t)$, and manage the loans $L(t)$. Thus, we introduce **liquidity constraint** (quantitative theory of money [32])

$$\tau_j pJ(t) + \tau_t Tax(t) + \tau_l L(t) \leq N(t). \tag{6.5}$$

This restriction means that payments from the current account are made in finite portions, so that, in fact, the average stock is proportional to the average flow, and the parameters τ_j, τ_t, and τ_l with dimension of time define characteristic times of money circulation in corresponding segments of the money system.

The result of identification of the constraint (6.5) is shown in Fig. 6.2.2. The parameters of identification are the following: $\tau_j = 0.06$, $\tau_t = 0.41$, $\tau_l = 0.28$.

Bank processes Producer's payments to other agents (and to Producer herself, since Producer is a macroagent) from the current account through Bank's correspondent account in CB. The latter in-

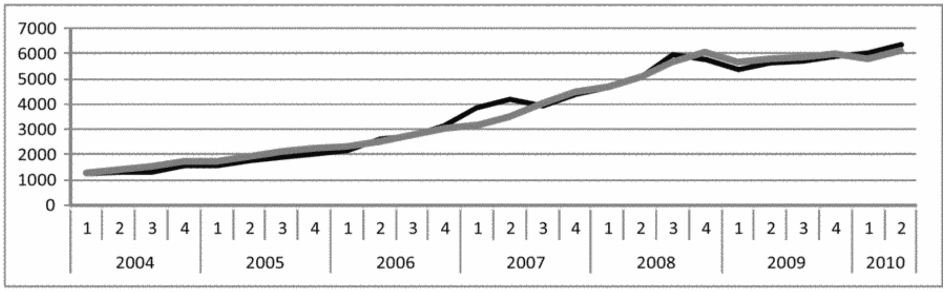

Fig. 6.2.2. Liquidity constraint (6.5). The grey line is the left-hand side and the black line is the right-hand side of the inequality(6.5).

strument is regarded as the universal means of payment in the model. Thus, when $N(t)$ is changed by payments or incoming transfers:

$$\frac{d}{dt} N(t) = NdN(t), \tag{6.6}$$

Bank posts an entry with a corresponding sum $KdN(t)$ in Bank's correspondent accounts in CB: $NdN(t) = KdN(t)$.

This sum contains all current payments and incoming transfers of Producer: revenue $pY(t)$, net loans $WdJ(t)$, expenditure on salaries $sR(t)$, interest payments $rL(t)$, tax payments $Tax(t)$, subsidies income $Sub_x(t)$ defined by government, investment spending $pJ(t)$, and dividends $Z(t)$. Producer does not have a current account; thus

$$0 = pY(t) - sR(t) - rL(t) + WdL(t) - Z(t) - Tax(t) + Sub_x(t) - pJ(t) - KdN(t). \tag{6.7}$$

We model tax payments of producer $Tax(t)$ using **effective** value-added tax rate ty and effective rate of social tax tes (see Sec. 6.7 for more details):

$$Tax(t) = ty \cdot pY(t) + tes \cdot sR(t). \tag{6.8}$$

Dividends $Z(t)$, as described in Sec. 3.5, are determined by a time-varying proportion $Ub_{up}(t)$, defined by Owner, which Producer learns from the block IA **up**.

$$Z(t) = Ud_{up}(t)\,\theta, \tag{6.9}$$

where θ is equity price owned by Producer.

This concludes the description of Producer's constraints. In canonical form of the model they are divided into groups: technological constraints **Tech**, for example, (6.1) and (6.2), which contain only agent's planned variables (with no indexes); groups of institutional constraints **Role**, for example (6.3), which contain not only planned, but also informational variables (with an IA index); and groups of balances **Balance**, for example, (6.4) or (6.7), which are specified in ECOMOD system by relations, their dimensions, and types of assets (*loans, current account*, etc).

6.2.2. Formulation and solution of Producer's optimal control problem. Producer's behavior is described by the problem of maximization of the equity price θ (see (3.23)) over all planned variables with respect to the constraints listed above. But before formulating and solving the problem, it is convenient to eliminate such "redundant" variables as $pJ(t)$ in (6.3), which were introduced in the model to have balances in an "easy-to-verify" form. ECOMOD processes this automatically with a special algorithm that recognizes redundant variables. The procedure reduces the Producer's system to the form[28]

$$0 = J(t) - \kappa M(t) - \frac{d}{dt} M(t), \tag{6.10}$$

$$0 = K(t) - \beta_l(t)L(t) - \sigma L(t) - \frac{d}{dt} L(t),$$

[28]The initial form of constraints is saved in the system for checking the balances after assembling the model.

$$\{0 \le K(t)\},$$

$$\big\{0 \le N(t) - \tau_j p_j(t)\big(J(t) - J_j(t)\big) - \tau_l L(t) - \tau_{tax}\big(ty\, p_y(t)\big(AM(t) + Be^{b(t-t0)}R(t)\big) + tes\, s_r(t)\, R(t)\big)\big\},$$

$$0 = p_y(t)(AM)(t) + Be^{b(t-t0)}R(t)) - s_r(t)R(t) + K(t) - \beta_l(t)L(t) - r_l(t)L(t) - Z(t)$$
$$- ty\, p_y(t)\big(AM(t) + Be^{b(t-t0)}R(t)\big) - tes\, s_r(t)R(t) - p_j(t)\big(J(t) - J_j(t)\big) - \frac{d}{dt}N(t) + Sub_x(t) \quad (6.11)$$

for the remaining planned variables $J(t)$, $K(t)$, $L(t)$, $M(t)$, $N(t)$, and $R(t)$.

As we can see, the system (6.10)–(6.11) has a form close to the typical form (3.1)–(3.3). Following the framework of Sec. 3, Producer's money is current accounts, and Eq. (6.11) describes the balance of this money (3.1). According to Sec. 3.4, we add a linear boundary condition on the growth of a linear combination of phase variables to (6.10)–(6.11)

$$\big(aL(t0)L(t0) + aM(t0)M(t0) + N(t0)\big)e^{\gamma(T-t0)} \le aL(T)L(T) + aM(T)M(T) + N(T). \quad (6.12)$$

ECOMOD automatically adds dual variables $\psi 1(t)$, $\psi 2(t)$, $\phi 3(t)$, etc., to the restrictions (6.10)–(6.12) in the problem $\theta \to \max$, writes down the Lagrange functional, and varies it with respect to the primal and the dual variables (see (3.7)). The result is the system of sufficient optimality conditions, similar to (3.8)–(3.10). For Producer it consists of

- three complementary slackness conditions, that correspond to the terminal constraint and the inequalities on the current values of the planned variables:

$$[\Phi\, 1]\,\big[aL(T)L(T) + aM(T)M(T) + N(T) - \big(aL(t0)L(t0) + aM(t0)M(t0) + N(t0)\big)e^{\gamma(T-t0)}\big], \quad (6.13)$$

$$[\phi 4(t)]\,\big[N(t) - \tau_j p_j(t)\big(J(t) - J_j(t)\big) - \tau_j L(t) - \tau_{tax}\big(ty\, p_y(t)\big(AM(t) + Be^{b(t-t0)}R(t)\big) + tes\, s_r(R(t))\big)\big],$$

$$[f3(t)]\,[K(t)],$$

- three dynamic equations of the original system:

$$0 = J(t) - \kappa\, M(t) - \frac{d}{dt}M(t),$$

$$0 = K(t) - \beta_l(t)L(t) - \sigma\, L(t) - \frac{d}{dt}L(t),$$

$$0 = p_y(t)\big(AM(t) + Be^{b(t-t0)}R(t)\big) - s_r(t)R(t) + K(t) - \beta_l(t)L(t) - r_l(t)L(t) - Z(t)$$
$$- ty\, p_y(t)\big(AM(t) + Be^{b(t-t0)}R(t)\big) - tes\, s_r(t)R(t) - p_j(t)\big(J(t) - J_j(t)\big) - \frac{d}{dt}N(t) + Sub_x(t),$$

- six optimality conditions, derived by varying the current values of planned variables:

$$0 = \big(-\psi 5(t)\, tes \cdot s_r(t) + Be^{bt0}\psi 5(t)p_y(t)\, e^{bt}$$
$$- ty\, Be^{-bt0}\psi 5(t)\, p_y(t)e^{bt} - \tau_{tax} ty\, Be^{-bt0}\phi 4(t)\, p_y(t)\, e^{bt} - \psi 5(t)\, s_r(t) - \phi 4(t)\, \tau_{tax}\, tes\, s_r(t)\big)\, dR,$$

$$0 = \left(\frac{d}{dt}\psi 5(t) + \phi 4(t)\right) dN,$$

$$0 = \left(\frac{d}{dt}\psi 1(t) + \psi 5(t)\, p_y(t)A - \phi 4(t)\tau_{tax} ty\, p_y(t)A - \psi 1(t)\kappa - \psi 5(t)\, ty\, p_y(t)A\right) dM,$$

$$0 = \left(\frac{d}{dt}\psi 2(t) - \phi 4(t)\tau_l - \psi 5(t)\beta_l(t) - \psi 5(t)r_l(t) - \psi 2(t)\beta_l(t) - \psi 2(t)\sigma\right) dL,$$

$$0 = \big(\phi 3(t) + \psi 2(t) + \psi 5(t)\big)\, dK,$$

$$0 = \big(\psi 1(t) - \phi 4(t)\tau_j p_j(t) - \psi 5(t)p_j(t)\big)\, dJ,$$

- three optimality conditions, derived by varying the terminal values of the phase variables:

$$\begin{aligned} 0 &= \left(-\psi 2(T) + \Phi 1 a L(T)\right) dL(T), \\ 0 &= \left(-\psi 1(T) + \Phi 1 a M(T)\right) dM(T), \\ 0 &= \left(-\psi 5(T) + \Phi 1\right) dN(T). \end{aligned} \tag{6.14}$$

Formal multipliers of the form dK and $dL(T)$ are added automatically into the optimality conditions for clarity and show which variable was varied to derive this condition. There is also an unknown variable θ in the problem. In order to find it, we need to take into account the expression (6.9) for the dividends and add to (6.13) and (6.14) a condition that the derivative of the Lagrange functional with respect to θ is equal to zero. However, we can do without this cumbersome condition.

In order to describe the interaction with Owner we need to specify the expression (3.18) and Eq. (3.19) for Producer's capital. ECOMOD also derives them automatically:

$$\begin{aligned} \Omega(t) &= \frac{\psi 1(t) M(t)}{\psi 5(t)} + \frac{\psi 2(t) L(t)}{\psi 5(t)} + N(t), \\ \frac{d}{dt}\,\Omega(t) &= \frac{\phi 4(t) \tau_j p_j(t) J_j(t)}{\psi 5(t)} + p_j(t) J_j(t) - Z(t) + Sub_x(t) - \frac{\Omega(t) \frac{d}{dt}\psi 5(t)}{\psi 5(t)}. \end{aligned} \tag{6.15}$$

These equations are more complicated than (3.19), because, as it is easy to verify, the Producer's problem violates the homogeneity symmetry more seriously than a typical problem (3.1)–(3.4). Nevertheless, the equation for capital is still linear and may be used just like (3.19). In particular, the Producer's problem is solvable if and only if the complementary slackness condition (6.14) is reduced to the equality

$$0 = \Omega(T) - \Omega(t0)\, e^{\gamma(T - t0)}.$$

After that, ECOMOD automatically proceeds to normalized dual variables (3.20) (saving the names) and returns (3.17). Special commands that adapt Maple functions to ECOMOD allow a researcher to exclude the dual variables where possible and introduce new variables to simplify the relations. Finally, the system automatically checks the dimensions and assigns an index J to the Producer's planned variables.

6.2.3. Producer's behavior. The system of equations in this section, like analogous systems from the following sections, does not contain expressions for excluded "redundant" variables. It is worth mentioning that they are not excluded completely, because they will appear again in interactions' descriptions in 6.11. The variables of the agent that are under control are described in Table 1.

$$\begin{aligned} &[\rho_J(t)][-ty\, R_J(t)\, Bp_y(t)], \tau_{tax} e^{bt - bt0} - \tau_{tax} tes\, s_r(t)\, R_J(t) \\ &\qquad - \tau_j p_j(t)\left(J_J(t) - J_j(t)\right) + N_J(t) - \tau_{tax} ty\, p_y(t)\, AM_J(t) - \tau_l L_J(t)], \end{aligned} \tag{6.16}$$

$$[-\psi 2_J(t) - 1]\,[K_J(t)], \tag{6.17}$$

$$\frac{d}{dt}\,p_j(t) = \iota_J(t)\, p_j(t), \quad \frac{d}{dt}\,M_J(t) = J_J(t) - \kappa M_J(t),$$

$$\frac{d}{dt} L_J(t) = \left(-\beta_l(t) - \sigma\right) L_J(t) + K_J(t),$$

$$\begin{aligned} &\frac{d}{dt}\,N_J(t) = Sub_x(t) - tes\, s_r(t)\, R_J(t) - s_r(t)\, R_J(t) + K_J(t) - Ub_{up}(t)\,\theta_J \\ &- R_J(t)\, Bp_y(t)(ty - 1)\, e^{bt - bt0} + \left(J_j(t) - J_J(t)\right) p_j(t) - p_y(t) A(ty - 1)\, M_J(t) + \left(-\beta_l(t) - r_l(t)\right) L_J(t), \end{aligned}$$

$$s_r(t) = -\frac{Bp_y(t)\, e^{b(t - t0)}(ty - 1 + ty\, \rho_J(t)\, \tau_{tax})}{tes + 1 + \rho_J(t)\, \tau_{tax} tes},$$

$$\frac{d}{dt}\,\rho_J(t) = -\frac{-\kappa + \iota_J(t)}{\tau_j} + \frac{p_y(t) A(ty - 1)}{\tau_j p_j(t)} + \left(\rho_j(t)\right)^2 + \left(-\frac{-\kappa \tau_j - 1 + \iota_J(t)\, \tau_j}{\tau_j} + \frac{ty\, Ap_y(t)\, \tau_{tax}}{\tau_j p_j(t)}\right) \rho_J(t),$$

Table 1. Producer's planned variables.

Name	Dimension	Meaning
$J_J(t)$	Product/time	Investments in fixed capital
$K_J(t)$	Money/time	Newly issued loans from Bank
$L_J(t)$	Money	Loan portfolio
$M_J(t)$	Product	Real fixed capital
$N_J(t)$	Money	Current accounts residuals
$R_J(t)$	Labor/time	Labor flow
$Y_J(t)$	Product	Output (real GDP)
$Z_J(t)$	Money/time	Dividends flow
$f_J(t)$	Money/time	Capital equation coefficients, which Producer transmits to Owner
$\nu_J(t)$	1	
$\rho_J(t)$	1/time	
$\psi2_J(t)$	1	Dual variable
$\Omega_J(t)$	Money	Producer's capital
$\iota_J(t)$	1/time	Gross investments inflation

$$\frac{d}{dt}\,\psi2_J(t) = (\beta_l(t)+\sigma)\,\psi2_J(t) + \beta_l(t) + r_l(t) + \big(\tau_l + \psi2_J(t)\big)\,\rho_J(t),$$

$$\Omega_J(t) = \big(p_j(t) + \rho_J(t)\,\tau_j p_j(t)\big)\,M_J(t) + N_J(t) + \psi2_J(t)L_J(t),$$

$$PP_J(t) = Ub_{up}(t)\,\theta_J,$$

$$0 = \Omega_J(T) - \Omega_J(t0)\,e^{\gamma(T-t0)}, \tag{6.18}$$

$$\nu_J(t) = 1, \quad f_J(t) = p_j(t)J_j(t) + \rho_J(t)\tau_j p_j(t)\,J_j(t) + Sub_x(t). \tag{6.19}$$

6.3. Macroagent Bank B.

6.3.1. Bank's constraints and the optimal control problem. In our model, Bank is a composition of commercial financial institutions, with material costs of financial transactions assumed to be negligible. The model of Bank used here was substantiated and studied separately in [3, 4].

In the model, Bank

- provides loans to Producer (IA **l**);
- attracts deposits from Household and nonresidents (IA **s**)
- attracts deposits from CB (IA **cb**);
- processes noncash transactions of Producer through Bank's correspondent account in CB (IA **cb**) and uses residuals of current accounts of Producer in Bank as a free credit resource (IA **n**);
- fulfils reserve requirements of the Central Bank on the residual of correspondent accounts and own capital
- pays dividends to Owner (IA **ub**).

Bank provides loans to Producer $L(t)$ with interest rate $r_l(t)$ for a term $\beta_l^{-1}(t)$:

$$\frac{d}{dt}\,L(t) = K(t) - HL(t) - ML(t), \quad \{0 \le K(t)\}, \tag{6.20}$$

$$ML(t) = \sigma L(t), \quad HL(t) = \beta(t)\,L(t), \tag{6.21}$$

$$WdL(t) = K(t) = HL(t), \quad rL(t) = r_l(t)\,L(t), \tag{6.22}$$

where $K(t)$ is newly issued loans, $HL(t)$ is loan repayment, $ML(t)$ is debt write-off, which is assumed to be proportional to the volume of loans, $WdL(t)$ is a cash flow of balance of loans and repayments, and $rL(t)$ is interest payments.

Although the relations (6.20)–(6.22) are identical in form to the relations (6.4), they have the opposite meaning. It is because in the process of assembly of the model the variables from (6.4) of "Producer's" block will be assigned an index J and will serve as **demand** for loans, and the variables from relations (6.20)–(6.22) will be assigned an index B and will serve as **supply** of loans. The relation $L_J(t) = L_B(t)$ in IA **l** will turn into a nontrivial equilibrium condition defining interest rate $r_l(t)$. The forms (6.4) and (6.20)–(6.22) coincide because both agents, Producer and Bank, use the same rules for credit operations.

Bank attracts deposits of Household and nonresident $S(t)$ on the same terms: interest rate is $r_s(t)$ and average maturity is $\beta_s(t)^{-1}$. This duration is assumed to be exogenous, and its dependence on time is again essential. The mechanism of attracting deposits that we describe is symmetrical to that of issuing loans, but without write-offs:

$$\begin{gathered}\frac{d}{dt}\,S(t) = V(t) - HV(t), \quad \{0 \le V(t)\},\\ WdS(t) = V(t) - HV(t), \quad HV(y) = \beta_s(t)\,S(t), \quad rS(t) = r_s\,S(t),\end{gathered} \tag{6.23}$$

where $V(t)$ is newly acquired deposits, $HV(t)$ is repayment of deposits, $WdS(t)$ is net inflow of deposits, and $rS(t)$ is interest payments on deposits.

In addition to deposits, Bank also attracts perpetual interest-free accounts $N(t)$. Because there is no interest that would regulate the supply, Bank has to rely on a forecast of the current accounts $N_n(t)$. In the extreme case, Bank can refuse to operate the account:

$$\frac{d}{dt}\,N(t) = dN(t), \quad \{N(t) \le N_n(t)\}. \tag{6.24}$$

A change in account means the client makes some payment through Bank's current account, whereby the current account changes by

$$WdN(t) = dN(t).$$

Bank also lends to and borrows from CB. In statistical data we see both periods when the CB deposits exceed CB loans and vice versa. Thus instead of modelling the CB deposits and loans separately, we model the net loans and deposits. The balance of CB loans and deposits is denoted by $Lc(t)$. The informational variables $r_{lcb}(t)$ and $r_{scb}(t)$ are interest rates of loans and deposits of CB. A change in the sign of the balance $Lc(t)$ leads to a change of interest rate. These changes are equivalent to two conditions on interest payments $rLc(t)$:

$$\{rLc(t) \le rl_{cb}(t)\,Lc(t)\}, \quad \{rLc(t) \le rs_{cb}(t)\,Lc(t)\}. \tag{6.25}$$

Operations with the Central Bank are assumed to be termless:

$$\frac{d}{dt}\,Lc(t) = dLc(t), \tag{6.26}$$

and the change of Bank's correspondent account due to these operations is

$$KLc(t) = dLc(t) - rLc(t).$$

Bank owns foreign currency assets in the amount of $Qw(t)$, and its dynamics is defined by expressions

$$\frac{d}{dt}\,Qw(t) = dQ(t), \quad \{0 \le Qw(t)\}.$$

Bank purchases foreign currency at the currency market (IA **w**), where Bank spends

$$WdQ(t) = dQ(t)\,w_w(t),$$

on purchase of foreign currency, where $w_w(t)$ is the exchange rate.

All the operations above Bank carries out through the correspondent account, whose balance is denoted by $A(t)$:

$$\frac{d}{dt}A(t) = WdS(t) - rS(t) - WdL(t) + rL(t) + WdN(t) - WdQ(t) - KLc(t) - Z(t),$$

where $Z(t)$ is the flow of dividends to Owner.

The Central Bank as the controlling institution requires commercial banks to fulfil certain regulations. The most important are **reserve requirements** and **capital sufficiency requirements**. The reserve requirement prescribes Bank to hold a fraction $Rc(t)$ of Bank's attracted funds on its correspondent account

$$Rc(t) = \zeta_{cb}(t)\big(S(t) + N(t)\big), \quad A(t) \geq Rc(t), \tag{6.27}$$

where $\zeta_{cb}(t)$ is the **reserve ratio**, which is assumed to vary over time.

The capital sufficiency condition prescribes the Bank's own capital to cover part of loans in the case of write-off:

$$\{\chi L(t) \leq A(t) + w_w(t)\,Qw(t) + L(t) + Lc(t) - S(t) - N(t)\}. \tag{6.28}$$

Here in the right-hand side we have an accounting valuation of the net assets of the bank. The capital sufficiency ratio χ, unlike reserve ratio, is assumed to be constant, because in (6.21) we assumed, somewhat voluntarily, the rate of failed debts write-offs σ to be constant.

The two last restrictions have special origin. Econometric study of Russian banking sector, the results of which are shown in [4], revealed two rather precise and stable dependencies (cointegrations [2]):

$$A(t) - Rc(t) \approx \tau\beta_s(t)\,S(t) + m\big(N(t) - Lc(t) - w_w(t)\,Qw(t)\big),$$
$$S(t) + N(t) \approx L(t) + Lc(t) + Rc(t).$$

Fit of the second equality is illustrated in Fig. 6.3.1.

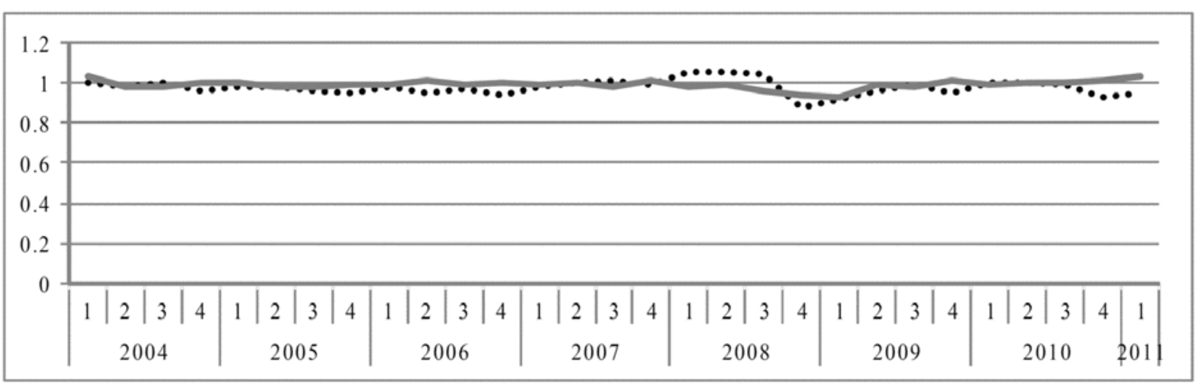

Fig. 6.3.1. The observed dynamics of the ratio attracted (deposits, current accounts) and issued (loans) funds $E(t)$. Dotted line—statistical data; solid line—calculation.

Although these dependencies are not based on any real requirements, we consider them as naturally established institutions specific for Russia. Unfortunately, **these equations cannot be used as exact ones for expression of planned variables**, because they poorly describe *increments* of the variables appearing in them, and in the model we use exactly the increments (see (6.23)). That is why we consider these empirical equations as realizations of some institutional or technological inequality constraints.

Since $W(t) = A(t) - Rc(t)$ is an unrestricted residual on correspondent account, the first discovered relation may be interpreted as Bank's need for liquidity. Thus we introduce the following inequality in our model:

$$\tau\beta_s(t)\,S(t) + m\big(N(t) - Lc(t) - w_w(t)\,Qw(t)\big) \leq A(t) - Rc(t). \tag{6.29}$$

It is harder to interpret the second relation, especially because it performs much better for the banking system as a whole than for any individual bank, including the biggest ones [4]. As was shown in [4], one interpretation is that the Russian banking system as a whole makes almost no credit emission, but prefers borrowing money abroad. We consider this constraint as a spontaneously established institution:

$$\{L(t) + Lc(t) + Rc(t) \leq S(t) + N(t)\}. \tag{6.30}$$

Due to the reasons given in Sec. 6.7.2, tax payments of Bank are not considered.

Bank as well as Producer acts in the interests of Owner based on the time ratio of dividends $Ub_{ub}(t)$, dividend payments are

$$Z(t) = \theta Ub_{ub}(t), \tag{6.31}$$

and solves the problem of maximization of the rate θ by choosing all planned variables (which are not indexed yet) under the constraints (6.20)–(6.31). As in the problem (6.12), we add to these constraints a terminal condition of linear form on the growth of phase variables:

$$\begin{aligned} 0 &\leq aL(T)\,L(T) + aLc(T)\,Lc(T) + aN(T)\,N(T) + aQw(T)\,Qw(T) + aS(T)\,S(T) + A(T) \\ &- \big(aL(t0)\,L(t0) + aLc(t0)\,Lc(t0) + aN(t0)\,N(t0) + aQw(t0)\,Qw(t0) + aS(t0)\,S(t0) + A(t0)\big)e^{\gamma(T-t0)}. \end{aligned}$$

The ECOMOD system searches for the solution under the same scheme as used in description of Producer's behavior (Sec. 6.2.2). The full system of sufficient conditions is given in [3]. The equation and the expression for own capital are deduced from the sufficient conditions (see (3.19), (3.18)). This theoretical expression (see below) is a linear combination of assets, similar to the right hand side of (6.28), but it contains coefficients, "corrected" for the dual estimates of these assets.

6.3.2. Solution and relaxation of complementarity slackness conditions. This section contains a concrete example of Bank behavior and a method of treatment of *complementarity slackness conditions* (**CSC**) (3.10). Recall that they appear because of inequality constraints. For example, the inequality in (6.24) leads to a complementarity slackness condition (CSC)

$$[\phi 6(t)][N_n(t) - N(t)]. \tag{6.32}$$

According to the principle of rational expectations the forecast of the balance of current accounts $N_n(t)$ should be realized. However, if Bank refuses to accept them ($N_n(t) > N(t)$), then the system of accounts will become mismatched. It means the equilibrium conditions are such that $\varphi 6(t) > 0$, so $N(t) = N_n(t)$. In the same way we **resolve** the CSC (6.32).

But for other inequities this simple reasoning does not fit. Statistics and past research experience of such models show that the CSC $[\phi 5(t)][V(t)]$, obtained from the inequality in (6.23), could be solved, but in an almost opposite way. The argument is as follows. Inflow of deposits $V(t)$ is positive and depends on incomes of Household rather than on the economic situation. Therefore we believe that

$$\varphi 5(t) = 0, \quad V(t) > 0.$$

We do not resolve other CSC, but *relax* (regularize) them. For example, consider an inequality in (6.20), that gives the CSC

$$[\phi 2(t)][K(t)]. \tag{6.33}$$

Computations show that this CSC is not easy to resolve, and we will use a different approach. From the economic point of view, CSC could be treated as **infinitely elastic functions of demand/supply of an agent**. Recall that dual variables indicate some internal evaluations. For example the relation (6.33) shows that Bank will offer new loans only if the internal price of these loans $\phi 2(t)$ is sufficiently low.

From the other point of view, the optimization problem we solve is linear, so that in a typical case its solutions are "corner solutions." If control variables are not bounded as, for example, $K(t)$, then the problem will be solvable (and not uniquely) only for a certain combination of exogenous variables. It seems that it becomes necessary to complicate the original problem by adding some nonlinearity. But we believe it is not too late to do this after derivation of optimality conditions for the original, simpler problem.

We assume that the CSC solves one of the main questions of economic theory: What combination of informational variables should be an argument of a demand/supply function? We will select a concrete form of this function empirically.

In other words, we replace the remaining CSC $[a]\cdot[b]$ by expressions $b = f(a)$, where $f(\cdot)$ is a monotonically decreasing function with a suitable normalization to be identified from empirical data. In the same manner we deal with the remaining unresolved CSC in other blocks. This appears to be an acceptable scheme, except for the inequality (6.30), the CSC for which can be written as

$$[\phi 3(t)][1 - E(t)], \quad E(t) = \frac{L(t) + Lc(t) + Rc(t)}{P(t)}, \quad P(t) = S(t) + N(t).$$

The point is that the value of $E(t)$ in fact appears in many inequalities, and by relaxing the CSC, we should take into account its "influence" on many boundaries. As a result, after exclusion of dual variables we obtain the expression

$$E(t) = e1 + e2 + e3\,\zeta_{cb}(t) + e3 + \left(-\frac{(e3{+}e1m)Liq(t)}{m} + (\chi e1 + e3)\,L(t) + \frac{e3(\tau\beta_s(t){-}m)\,S(t)}{m}\right)\big(P(t)\big)^{-1},$$
$$Liq = A(t) - Rc(t) + w_w(t)Qw(t),$$

where the new constant coefficients $e1$, $e2$, and $e3$ appear when we relax the CSC. Figure 6.3.1 shows how this formula with suitable choice of the coefficients reproduces the observed value of $E(t)$ by substituting statistical data for all values on the right hand side.

6.3.3. Bank's behavior. After relaxing the CSC we obtain the following system of equations:

$$\begin{aligned} rLc_B(t) = {} & \frac{\begin{pmatrix}(-arl2 + brl2)brl1 - arl1\,brl2 + brl1\,rl_{cb}(t)(\zeta_{cb}(t))^2 \\ + ((-rl_{cb}(t) + arl2 - brl2)brl1 - rs_{cb}(t)brl2)\zeta_{cb}(t)\end{pmatrix} P(t)}{-brl1 + brl1\zeta_{cb}(t) - brl2} \\ & + \frac{P(t)(-rs_{cb}(t)brl2 + brl1\,rl_{cb}(t)\zeta_{cb}(t) - rl_{cb}(t)brl1)\,E(t)}{-brl1 + brl1\zeta_{cb}(t) - brl2} \\ & - \frac{(-rs_{cb}(t)brl2 + brl1rl_{cb}(t)\zeta_{cb}(t) - rl_{cb}(t)brl1)\,L_B(t)}{-brl1 + brl1\zeta_{cb}(t) - brl2} \qquad (6.34) \\ & = \Omega_B(T) - \Omega_B(t0)\,e^{\gamma(T-t0)}, \qquad (6.35) \end{aligned}$$

Table 2. Projected variables of Bank.

Name	Dimension	Meaning
$E_B(t)$	1	Ratio of attracted and issued funds
$Liq_B(t)$	Money	Bank liquidity
$P_B(t)$	Money	Attracted funds
$A_B(t)$	Money	Residuals on correspondent accounts in the Central Bank
$K_B(t)$	Money/time	Newly issued loans
$L_B(t)$	Money	Loans to Producer
$Lc_B(t)$	Money	Net deposits in the Central Bank
$S_B(t)$	Money	Deposits
$Qw_B(t)$	Currency	Capital of Bank
$F_B(t)$	Money/time	The coefficients in the equation for capital, that Producer transmits to Owner
$\nu_B(t)$	1	
$\rho_B(t)$	1/time	

$$\nu_B(t) = 1,$$

$$f_B(t) = \frac{-rl_{cb}(t)N_n(t) + rl_{cb}(t)N_n(t)\zeta_{cb}(t) + rl_{cb}(t)N_n(t)\,m}{-1+m}$$
$$+ N_n(t)\zeta_{cb}(t)\left(\frac{\begin{array}{c}-rl_{cb}(t)\zeta_{cb}(t) + rl_{cb}(t) - r_s(t) + r_s(t)m \\ -\beta_s(t) + \beta_s(t)m - rl_{cb}(t)\tau\beta_s(t)\end{array}}{-1+m} - \beta_s(t)\right) m(m\zeta_{cb}(t) - m + \tau\beta_s(t))^{-1},$$

$$PP_B(t) = Ub_{ub}(t)\theta_B,$$

$$\rho_B(t) = \frac{m + aliq - 1 + m\zeta_{cb}(t) - arho}{(-1+m)brho} - (brho + bliq)$$
$$\times\left(\frac{-rl_{cb}(t)\zeta_{cb}(t) + rl_{cb}(t) + rl_{cb}(t)m\zeta_{cb}(t) - rl_{cb}(t)m}{m\zeta_{cb}(t) - m + \tau\beta_s(t)} + \frac{(-1+m)r_s(t)}{m\zeta_{cb}(t) - m + \tau\beta_s(t)}\right)(-1+m)^{-1}brho^{-1}$$
$$- \frac{E(t)}{brho} + \left(-\frac{(-m + chiB)L_B(t)}{(-1+m)brho} + \frac{(\tau\beta_s(t) - m)S_B(t)}{(-1+m)brho} - \frac{Q_B(t)}{brho}\right)\big(P(t)\big)^{-1},$$

$$Liq(t) = mL_B(t) + (\tau\beta_s(t) - m)S_B(t) + (1-m)Q_B(t) + \Bigg(aliq - mE(t) + m\zeta_{cb}(t) + m - bliq$$
$$\times\left(\frac{-rl_{cb}(t)\zeta_{cb}(t) + rl_{cb}(t) + rl_{cb}(t)m\zeta_{cb}(t) - rl_{cb}(t)m}{m\zeta_{cb}(t) - m + \tau\beta_s(t)} + \frac{(-1+m)r_s(t)}{m\zeta_{cb}(t) - m + \tau\beta_s(t)}\right)\Bigg)P(t),$$

$$Q_B(t) = \Bigg(aq - \frac{(\frac{d}{dt}w_w(t))bq(\zeta_{cb}(t) - 1)}{w_w(t)r_s(t)} + bq(\zeta_{cb}(t) - 1)$$
$$\times\left(\frac{-rl_{cb}(t)\zeta_{cb}(t) + rl_{cb}(t) + rl_{cb}(t)m\zeta_{cb}(t) - rl_{cb}(t)m}{m\zeta_{cb}(t) - m + \tau\beta_s(t)} + \frac{(-1+m)r_s(t)}{m\zeta_{cb}(t) - m + \tau\beta_s(t)}\right)\big(r_s(t)\big)^{-1}\Bigg)P(t),$$

$$E(t) = e1 + e2 + e3\zeta_{cb}(t) + e3$$
$$+ \left(-\frac{(e3 + e1\,m)Liq(t)}{m} + (chiBe1 + e3)\,L_B(t) + \frac{e3(\tau\beta_s(t) - m)S_B(t)}{m}\right)\big(P(t)\big)^{-1},$$

$$K_B(t) = \Bigg(\Bigg(-\frac{\rho_B(t)\,chiB}{\beta_l(t)+\rho_B(t)} + \frac{r_s(t)}{(\zeta_{cb}(t)-1)(\beta_l(t)+\rho_B(t))} + \frac{r_l(t)}{\beta_l(t)+\rho_B(t)} - \big(-chiB + chiB\zeta_{cb}(t) + \tau\beta_s(t)\big)$$
$$\times\left(\frac{-rl_{cb}(t)\zeta_{cb}(t) + rl_{cb}(t) + rl_{cb}(t)m\zeta_{cb}(t) - rl_{cb}(t)\,m}{m\zeta_{cb}(t) - m + \tau\beta_s(t)} + \frac{(-1+m)\,r_s(t)}{m\zeta_{cb}(t) - m + \tau\beta_s(t)}\right)$$
$$\times\big(\beta_l(t) + \rho_B(t)\big)^{-1}(-1+m)^{-1}(\zeta_{cb}(t) - 1)^{-1}\Bigg)bk + ak\Bigg)P(t),$$

$$0 = (-\beta_l(t) - \sigma)\,L_B(t) + K_B(t) - \frac{d}{dt}\,L_B(t),$$

$$Lc_B(t) = \big(E(t) - \zeta_{cb}(t)\big)\,P(t) - L_B(t),$$

$$Ub_{ub}(t) = \frac{-\frac{d}{dt}\,A_B(t) + V_B(t) + \frac{d}{dt}\,N_B(t) - K_B(t) + rLc_B(t) - \frac{d}{dt}\,Lc_B(t)}{\theta_B}$$
$$- \frac{(\beta_s(t) + r_s(t))\,S_B(t)}{\theta_B} + \left(\frac{\beta_l(t)}{\theta_B} + \frac{r_l(t)}{\theta_B}\right)L_B(t) - \frac{\frac{d}{dt}Q_B(t)}{\theta_B}$$
$$+ \frac{Q_B(t)\frac{d}{dt}w_w(t)}{\theta_B w_w(t)} + \left(\frac{L_B(t)}{\theta_B P(t)} - \frac{E(t)}{\theta_B} + \frac{\zeta_{cb}(t)}{\theta_B} + \frac{Lc_B(t)}{\theta_B P(t)}\right)\frac{d}{dt}\,P(t),$$

$$\Omega_B(t) = (E(t) - \zeta_{cb}(t) - 1)\,P(t) + Q_B(t) - \Bigg(-\sigma + r_l(t) - \frac{(-m + chiB)\,r_s(t)}{m\zeta_{cb}(t) - m + \tau\beta_s(t)} -$$

$$- \frac{(chiB + chiB\zeta_{cb}(t) + \tau\beta_s(t))\,rl_{cb}(t)}{m\zeta_{cb}(t) - m + \tau\beta_s(t)} - \rho_B(t)\,chiB\Bigg) L_B(t)\left(-\sigma - \beta_l(t) - \rho_B(t)\right)^{-1} + A_B(t). \qquad (6.36)$$

6.4. Macroagent Household H.

6.4.1. Constraints and the problem of maximization of expected utility of consumption. The block of Household, that represents aggregated households in the model,

- buys and consumes the product (IA **c**);
- sells labor and receives salary (IA **r**);
- receives transfers from the public budget (IA **x**);
- saves money in the Bank and receives interest income (IA **s**).

We neither found in literature, nor managed to invent a plausible model of the Russian labor market. As a result we have to describe employment as an exogenous variable in the model. It corresponds to an assumption that a household does not plan the amount of labor, but occupies job vacancies described by $R_r(t)$ and agrees to the market wage $s_r(t)$, so the total wage bill is given by

$$sR(t) = s_r(t)\,R_r(t). \qquad (6.37)$$

The standard macroeconomic description of the consumer demand based on current consumption utility does not reproduce the observed interrelation of the phases of consumption and inflation fluctuations. To account for this, we distinguish product purchases $C(t)$ from consumption $CC(t)$ that maximizes the utility of Household. Formally we assume that the purchased product is accumulated in the form of stock $Q(t)$, and consumption is analogous to "amortization" of this stock (similar to durable goods):

$$\frac{d}{dt}\,Q(t) = C(t) - CC(t), \quad CC(t) = \mu \cdot Q(t).$$

Purchases $C(t)$ are paid for the price $p_c(t)$, and the total consumer expenditure is

$$pC(t) = p_c(t)\,C(t).$$

Household saves $S(t)$ at interest $r_s(t)$ for a period of $\beta_s^{-1}(t)$:

$$\frac{d}{dt}\,S(t) = V(t) - H(t), \quad \{0 \le V(t)\},$$

$$WdS(t) = V(t) - H(t), \quad H(t) = \beta_s(t)\,S(t), \quad rS(t) = r_s(t)\,S(t),$$

where $V(t)$ is the flow of new deposits, $H(t)$ is withdrawals, $rS(t)$ is interest payments, and $WdS(t)$ is the financial result of transactions with deposits.

Household receives transfers $SB_x(t)$ from the public budget and pays with cash $A(t)$

$$\frac{d}{dt}\,A(t) = -WdS(t) + rS(t) - pC(t) + sR(t) + SB_x(t), \quad \{0 \le A(t)\}. \qquad (6.38)$$

Relations (6.37) and (6.38) are the main constraints for Household. The purpose of Household is traditional maximization of expected discounted utility of consumption [32]

$$\int_{t0}^{T} \frac{(CC(t)/(\mu Q0))^{1-\eta}}{1-\eta}\,e^{-\Delta t}\,dt,$$

where $Q0$, Δ, $\eta > 0$ are adjustment parameters.

Household's problem is solved according to the scheme described in Sec. 6.2. It differs from the typical problem (Sec. 3.1) because "the utility-bearing stream" $CC(t)$ is material, not financial. However, the Household's problem satisfies the desired homogeneity property, and all the arguments about the

Table 3. Projected variables of Household.

Name	Dimension	Meaning
$q(t)$	Time	Dual variable
$C_H(t)$	Product/time	Purchase of consumer product
$Q_H(t)$	Product	Stock of consumption good
$S_H(t)$	Money	Stock of deposits
$V_H(t)$	Money/time	Increase in deposits
$WdS_H(t)$	Money/time	Net deposits flow
$i_H(t)$	1/time	Consumer product price inflation
$A_H(t)$	Money	Stock of cash
$\nu_H(t)$	1	Dual variable
$\Omega_H(t)$	Money	Capital

capital apply here as well. Moreover, in this case we need only the expression for the capital in a terminal condition, but do not need the coefficients of equation for change of capital, since Household has no owner.

6.4.2. Household's behavior. After normalization and exclusion of dual variables, introduction of a new variable $q(t)$ instead of the return on capital $\rho(t)$, $\rho(t) = \big(q(t)\big)^{-1} - \mu + \iota(t)$, and indexing the planned variables the system of sufficient conditions of optimality turns to

$$\big[\big(q(t)\big)^{-1} - \mu + \iota_H(t)\big]\,[A_H(t)], \quad \frac{d}{dt}\,Q_H(t) = C_H(t) - \mu Q_H(t), \quad \frac{d}{dt}\,p_c(t) = \iota_H(t)\,p_c(t), \tag{6.39}$$

$$0 = \Omega_H(T) - \Omega_H(t0)\,e^{\gamma(T-t0)}, \tag{6.40}$$

$$[\nu_H(t)]\,[V_H(t)], \quad \frac{d}{dt}\,S_H(t) = V_H(t) - \beta_s(t)\,S_H(t),$$

$$\frac{d}{dt}\,A_H(t) = \big(r_s(t) + \beta_s(t)\big)\,S_H(t) - V_H(t) + SB_x - p_c(t)\,C_H(t) + s_r(t)\,R_r(t),$$

$$\frac{d}{dt}\,\nu_H(t) = \big(-\mu + \iota_H(t) + \beta_s(t)\big)\,\nu_H(t) + r_s(t) + \mu - \iota_H(t) + \frac{\nu_H(t) - 1}{q(t)},$$

$$\Omega_H(t) = (-\nu_H(t) + 1)\,S_H(t) + p_c(t)\,Q_H(t) + A_H(t),$$

$$\frac{d}{dt}\,q(t) = -1 + \left(\mu + \frac{\eta C_H(t)}{Q_H(t)} - \eta\mu + \Delta\right) q(t). \tag{6.41}$$

6.5. Macroagent Owner C.

6.5.1. Constraints. In intertemporal equilibrium models the functions of owner are usually assigned to households. But, as it was already noted at the end of Sec. 3.5, we separate owners of capital from households according to Russian realities. Macroagent Owner, described in the model, is an aggregate of individuals and entities, which control allocation of capital between sectors and abroad. The justification for this approach is the fact that in the previous, single-product version of the model we managed to describe correctly the nontrivial dynamics of capital import/export from the country by introduction of Owner [21].[29]

[29] It may seem strange that, in describing the movement of capital, we did not describe the stock market in the model. But in Russia it is (as well as in developed countries in 1980s), in the best case, "a barometer," evaluating stock prices, and allocation of capital is a result of direct transactions.

In the model, macroagents Producer, Bank, and Merchant belong to Owner. First, let us consider the interaction with Producer. According to the scheme described in Sec. 3.5, Owner receives forecast of coefficients of dynamic equation of capital **up** $\rho_J(t)$ and (6.19) and equity price θ_J (6.9) from Producer. According to this information, Owner makes a forecast of change of investment in the capital of the Producer $Kp(t)$ (see (3.25)) depending on the time proportion $zp(t)$ of receipt of dividends of Producer $Zp(t)$ set by Owner. Because Producer's problem is different from the standard one, the equation for $Kp(t)$ is more complicated than (3.25):

$$\frac{d}{dt}\,Kp(t) = \rho_{up}(t)\,Kp(t) - zp(t)\,\nu_{up}(t) + \frac{f_{up}(t)}{\theta_{up}}, \quad Kp(t) \geq 0, \quad Zp(t) = \theta_{up} zp(t). \tag{6.42}$$

One should bear in mind that axioms of canonical form prohibit using planned variables of one agent in the block describing another agent, which is why transfer of information, for example, about $\rho_J(t)$, is described explicitly (see (6.59)) as transformation of this value to the signal $\rho_{ub}(t)$ for Producer. Note that the differential equation (6.42) is not considered as a balance, or as a technological constraint, like (6.2), so its form has no restrictions.

In the same way, taking into account the obvious changes in notation we describe how Owner controls the Bank's capital:

$$\frac{d}{dt}\,Kb(t) = \rho_{ub}(t)\,Kb(t) - zb(t)\,\nu_{ub}(t) + \frac{f_{ub}(t)}{\theta_{ub}}, \quad 0 \leq Kb(t), \quad Zb(t) = \theta_{ub} zb(t). \tag{6.43}$$

Profit of Merchant $Zq_{uq}(t)$ is nonzero because of account receivables of public budget (see Sec. 6.6), and can be considered as an informational variable, that is, exogenous value for Owner.

Transfer of capital across borders is described as the purchase or currency $V(t)$ at the rate of $w_w(t)$ in the amount of $wV(t)$, which form Owner's foreign assets $Q(t)$. For simplicity, they are considered to be unprofitable, so

$$\frac{d}{dt}\,Q(t) = V(t), \quad wV(t) = w_w(t)\,V(t). \tag{6.44}$$

There is no requirement of nonnegativity of the stock $Q(t)$. Negative foreign assets are considered as long-term investments in the Russian economy. These investments differ from foreign deposits, passing through an agent Exporter (see Sec. 6.10).

It is assumed that Owner, in contrast to Household, does not purchase a consumer product. In our view, consumption expenditures of Owner are insignificant in Russia compared to flows from dividends and capital export. Besides, these expenditures have little effect on the behavior of Owner and on the capability of investing in capital. Owner seeks to maximize the property abroad, acquired by currency. Therefore, the financial balance of Owner has the following form:

$$\frac{d}{dt}\,A(t) = Zp(t) + Zb(t) + Zq_{uq}(t) - wV(t), \quad 0 \leq A(t).$$

6.5.2. Owner's problem and its solution. As a functional of Owner we consider

$$\int\limits_{t_0}^{T} \left(\frac{Q(t)}{Q_0}\right) e^{-\delta(t-t_0)}\,dt, \tag{6.45}$$

where Q_0 is a normalizing constant factor. This functional looks very similar to the discounted utility, but it is not so. The traditional interpretation of the discount rate δ as profitability of alternative investments is natural when one considers discounted *flow* of income or expenditure. Here we discount *stock*, and a natural interpretation of δ is **a measure of risk**. With a large planning horizon $T - t_0$, the value (6.45) can be considered as the expected value of stock of the currency $\mathbf{E}_u\{Q(u)\}$ at a *random* moment of time $u > t0$, having a probability density function $E^{-\delta(t-t_0)}du$. Owner's problem can thus be seen as a problem of optimization of risky investments $Kb(t)$ and $Kp(t)$, which have sufficiently high returns, but may be lost at a random moment of time u. The shape of the distribution shows that u is a Poisson point, and δ^{-1} is the

Table 4. Planned variables of Owner.

Name	Dimension	Meaning
$A_C(t)$	Money	Residuals on the Owner account
$E_C(t)$	1/time	Auxiliary variable
$Kb_C(t)$	1 (item)	Investments in the capital of Bank
$Kp_C(t)$	1 (item)	Investments in the capital of Producer

average time of standby (it is independent of l_0 due to the property of a Poisson process). Thus, the greater the value of δ, the greater the risk and the earlier, other things being equal, Owner wants to get the stock Q.

Integrating (6.45) by parts, taking into account Eq. (6.44), the functional of Owner is reduced to the form

$$\int_{t0}^{T} \frac{(\mathbf{e}^{(-\delta\, T+\delta\, t0)} + \mathbf{e}^{(-\delta\, t+\delta\, t0)})V(t)}{\delta\, Q0}\, dt. \tag{6.46}$$

This functional has the form (3.4). Therefore, the system of sufficient conditions can be derived by the same procedure as described in Sec. 6.2 if we add the terminal condition

$$\big(aKb(t0)\, Kb(t0) + aKp(t0)\, Kp(t0) + A(t0)\big)\, e^{\gamma(T-t0)} \leq aKb(T)\, Kb(T) + aKp(T)\, Kp(T) + A(T).$$

6.5.3. Description of Owner's behavior.

$$\left[-\rho_{up}(t) + \delta - E_C(t) + \frac{\frac{d}{dt}w_w(t)}{w_w(t)} + \frac{\frac{d}{dt}\nu_{up}(t)}{\nu_{up}(t)}\right][Kp_C(t)], \tag{6.47}$$

$$\left[-\rho_{ub}(t) + \delta - E_C(t) + \frac{\frac{d}{dt}w_w(t)}{w_w(t)} + \frac{\frac{d}{dt}\nu_{ub}(t)}{\nu_{ub}(t)}\right][Kb_C(t)],$$

$$E_C(t) = \frac{\delta\mathbf{e}^{-\delta(T-t0)}}{-\mathbf{e}^{-\delta(t-t0)} + \mathbf{e}^{-\delta(T-t0)}},$$

$$\frac{d}{dt}\, Kb_C(t) = \rho_{ub}(t)\, Kb_C(t) - zb_C(t)\, \nu_{ub}(t) + \frac{f_{ub}(t)}{\theta_{ub}} \tag{6.48}$$

$$\frac{d}{dt}\, Kp_C(t) = \rho_{up}(t)\, Kp_C(t) - zp_C(t)\, \nu_{up}(t) + \frac{f_{up}(t)}{\theta_{up}}, \tag{6.49}$$

$$0 = \theta_{up} zp_C(t) + \theta_{ub} zb_C(t) + Zq_{uq}(t) - w_w(t)V_C(t),$$

$$0 = \Omega_C(T) - \Omega_C(t0)\, e^{\gamma(T-t0)}, \tag{6.50}$$

$$\Omega_C(t) = \frac{\theta_{ub} Kb_C(t)}{\nu_{ub}(t)} + \frac{\theta_{up} Kp_C(t)}{\nu_{up}(t)}. \tag{6.51}$$

6.6. Macroagent Trader Q. As was already mentioned, this somewhat formal agent implements the product aggregation procedure described in Sec. 4. Trader purchases real GDP $Y(t)$ from Producer at the price $p_y(t)$ and "produces" export $E(t)$ and domestic $X(t)$ products from GDP according to the "technology" (4.12). Trader sells product $E(t)$ to Exporter at the price $p_e(t)$. Also Trader produces consumption product $C(t)$ according to (4.10) and investment product $J(t)$ according to (4.11) from domestic product $X(t)$ and import product $I(t)$ purchased from Importer at the price $p_i(t)$. Then Trader sells consumption product $C(t)$ to Household (as a consumer) and State (as a consumer, see Sec. 6.7). Also Trader sells investment product $J(t)$ to Producer (in role of investor) and State (in role of investor (6.3)). The goal of Trader is to maximize return $Z(t)$, which is determined from the financial balance

$$0 = p_e(t)\, E(t) + p_c(t)\, C(t) + p_j(t)\, J(t) - p_i(t)\, I(t) - p_y(t)\, Y(t) - Z(t). \tag{6.52}$$

This problem separates into the problems (4.15) and (4.19), which provide conditions of rationality. According to these conditions, the maximum profit should be zero. In fact, in the model it is nonzero and equal to the increase in credit debts of State (see Sec. 6.7.1), which systematically underpays for delivered consumer product.

Summing up, the description of Trader in the model consists of financial balance (6.52), material balance (4.9), aggregating expressions (4.10)–(4.12), and conditions of rationality (4.17), (4.18), and (4.20).

6.7. Individual Agent State G.

6.7.1. Description of State behavior. In the "State" block we mostly describe the Ministry of Finance execution of the consolidated budget. The consolidated budget balance combines balances of the federal budget and of all the regional budgets within Russian Federation, but does not include out-of-budget funds (such as pension fund, national health-care fund, and so on). Agent "State" performs the following functions in the model:

- receives budget income (IA **x**);
- pays subsidies to Producer (IA **x**);
- pays transfers to Household (IA **x**);
- purchases consumer product (IA **c**) for public consumption;
- purchases investment product (IA **j**) for public investment in fixed assets;
- bears credit debt for public consumption;
- uses the asset "corresponding account within the Central Bank" as a current account.

This traditional set of State's controls describes it as a "redundant consumer," where positive impact of government consumption on standard of living and economic growth can only be indirect. But essentially in an economy the state does not play a role of consumer but a role of producer of public goods, such as order, justice, security, guaranteed level of life, and the costs of public consumption and salaries of civil servants are paid as factors of this production. Also State makes in a sense no-return investments, such as construction of infrastructure facilities. Description of the State as a manufacturer would be more correct, but it significantly complicates the model. So far we do not describe the result of production activity of State explicitly in the model, and we take into account only the costs of these activities.

Since "State" is an individual agent in the model and its behavior is described not with the solution of the optimization problem, but with a scenario, and the difference of planned and informational variables becomes blurred, we omit its canonical form, and just comment on a simplified system of indexed relations.

Statistics show a significant budget debt, which is paid off at the end of each year, so we distinguish the product consumed by the state $G_G(t)$ and paid product $G_{-}p_G(t)$. They are given by the scenario. Product $G_{-}p(t)$ paid at price $p_c(t)$.

Product $G_G(t)$ includes not only the procurement of public consumption, but also the salary of public employees. In the national accounts State consumes and pays (at cost) the product of a special industry "Management," which makes this product through the raw capital and labor employed in the public sector. Consequently, in the model State employees are included in the total number of employees $R_J(t)$.

State buys investment product in the amount of $J_G(t)$ at the price $p_j(t)$.

Consolidated budget revenues are generated by taxes. State designates tax rates, and expects revenue. That is why the value of budget income is an informational variable $BudInc_x(t)$. It is defined as the amount of tax payments of all agents in IA **x**.

Expenses of the **consolidated budget** are divided into four categories. Firstly, there is public consumption $pG_{-}s(t)$, known from statistics [36]. Secondly, there is public investments $pJ_{-}s(t)$, included in the statistical gross investments $J(t)$ (see (6.3)). Thirdly, there are subsidies for Producer $Sub_{-}s(t)$. Fourthly, there are transfers to Household $Tr_{-}s(t)$ (pensions, scholarships, grants, etc., but without salaries of public employees). Salaries of public employees are a part of public consumption as the state "consumes" services of these employees.

Table 5. Planned variables of State defined with scenario.

Name	Dimension	Meaning
$G_G(t)$	Product/time	Actual consumption
$G_{-}p_G(t)$	Product/time	Paid State's consumption
$J_G(t)$	Product/time	State's investments
$Ng_G(t)$	Money	Current account within the Central Bank
$Tr_G(t)$	Money/time	Transfers to Household
$SubPr_G(t)$	Money/time	Subsidies to Producer
$BudInc_x$	Money/time	Tax revenue

Incomes are received and expenses are deducted from account of Ministry of Finance within the Central Bank (see Sec. 6.8). Current account balances are described by the variable $Ng(t)$. Increment of State's current accounts and its usage through correspondent accounts within the Central Bank is described in a way similar to that in (6.6), (6.7) and provide the dynamic equation of the account balances of the Treasury (the accumulated surplus of budget). Its balance sheet is the description of the State in the model:

$$0 = -\frac{d}{dt} Ng_G(t) + BudInc_x(t) - tj\, p_j(t)\, J_G(t) - p_c(t)\, G_{-}p_G(t) - p_j(t)\, J_G(t) - SubPr_G(t) - Tr_G(t). \quad (6.53)$$

This relation is the only constraint on the scenario of State's projected variables.

6.7.2. Econometric model of taxation. While tax revenues are formally defined in the block IA **x**, the problems that emerge in connection with taxation need separate discussion which we provide in this section. In the model of 2004 [7] and its modifications [21], budget income was determined by the statutory tax rate adjusted for shadow turnover. However, this model revealed deviations from the standards of income so large that they cannot be explained by tax evasion.

For example, Fig. 6.7.1 shows that for a statutory rate of 18 % the percentage of VAT revenues in GDP are at most 10 % and decrease. The same proportion is observed in the EU countries with their strict tax discipline, so that the observed effect hardly can be attributed to criminal tax evasion.

One has to accept the fact that increase in tax exemptions and extensive use of noncriminal tax optimization schemes led to a situation, when the macroeconomic tax rates and base become absolutely uninformative for estimation of budget income.

Moreover, the structure of tax revenues has changed. For example, mineral extraction tax (MET or severance tax) began to constitute a large share of the budget income, but we do not account for its base in our model. The only type of tax for which we found a suitable base among macroeconomic indicators

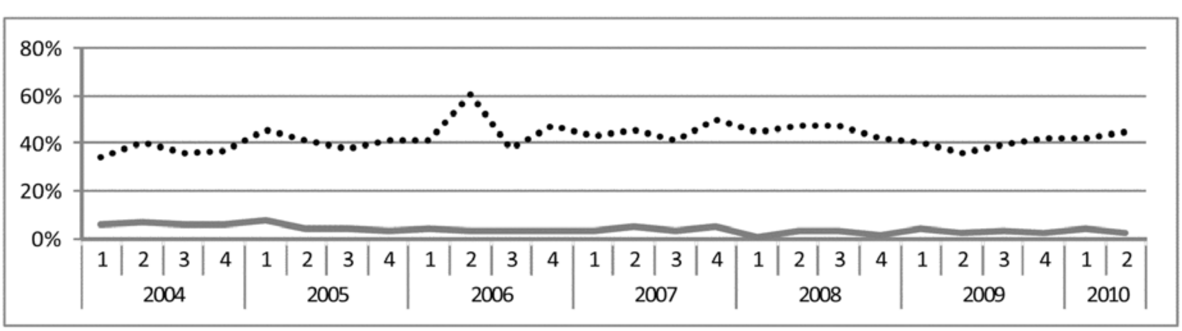

Fig. 6.7.1. Statistical analogies of tax rates. Dotted line is ratio of budget income to GDP; solid line is proportion of VAT revenues to GDP.

is the unified social tax (UST). Its revenues amount to an almost stable 4–5 % of the gross payroll given by statistical data:

$$\mathrm{UST}(t) = tes \cdot s_r(t)\, R_r(t), \quad tes = 0.046.$$

For the sum of tax and other revenues of budget (excluding import and export duties), which is mainly VAT, MET, tax on profits (TP), and the Income Tax (IT), we found and used in the model the following econometric relationship:

$$\begin{gathered}\text{Budget revenues} - \text{Duties} - \text{Unified social tax} = ty \cdot \mathrm{GDP} - te \cdot \mathrm{Export} + ti \cdot \mathrm{Import}, \\ ty = 0.344, \quad te = 0.170, \quad ti = 0.183.\end{gathered} \tag{6.54}$$

Accuracy of the expression (6.54) is characterized by Fig. 6.7.2. Although the series are nonstationary, the regression is correct, because the expression (6.54) is a good cointegration [2].

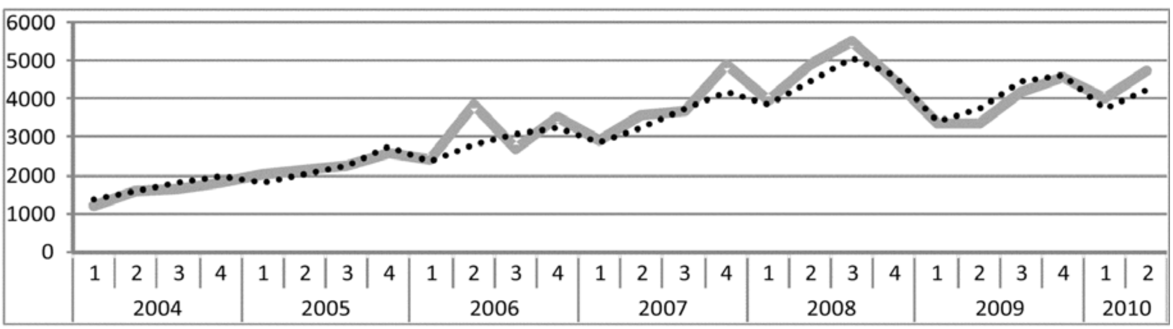

Fig. 6.7.2. Total income of the budget, where part of income net of duties and UST is calculated by(6.54). Solid line is statistical data, dotted line represents calculation.

The expression (6.54) is easy to interpret. At 18 % of VAT the term $-te \cdot \mathrm{Export}$ can be regarded as the value of the VAT refund for export, and the term $ti \cdot \mathrm{Import}$ as the value of the import tax. Then the remaining part on the right hand side describes the total revenues of VAT, PT, MET, and IT as some common tax burden on GDP. It is natural to impose it together with the UST on Producer as was done above (see (6.8)). The remaining terms on the right hand side of (6.54) we refer to Exporter and Importer, respectively (see below).

As for the rates of import duties $ni_x(t)$ and export duties $ni_e(t)$ we define it as simply a ratio of the corresponding payments to budget [35] to statistical volumes of exports and imports in current prices. We extrapolate them and use them as exogenous variables. The dynamics of effective duty is shown on Fig. 6.7.3.

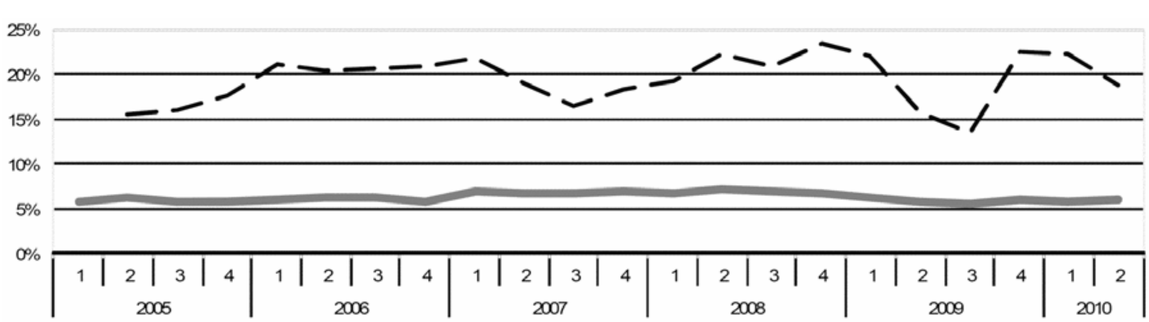

Fig. 6.7.3. Effective tax. The dashed line is the aggregate export duty rate. The solid line is the aggregate rate of import duties.

6.8. Individual Agent Central Bank CB. In Russia, like in the majority of countries, the Central Bank is a public agency responsible for functioning of monetary and credit systems. The main objectives of the Central Bank as a public authority are to ensure regularity in operation of accounts system, to keep inflation and exchange rate within the boundaries defined by the budget policy, and to ensure the stability of the banking system and credit accessibility. In addition to administration tools such as licensing of banks and bank reserve requirements, the Central Bank has specific economic policy instruments. As a commercial organization, the Central Bank is obliged to operate without loss. The law ensures independence of the Central Bank in its management of foreign exchange reserves and loans. That is why we describe CB in the model as a separate unit.

In the model Central Bank performs the following functions:

- holds reserves in assets;
- lends credits to the banking system and keeps banking system deposits (IA **cb**)[30]
- issues money, thereby forms monetary base M0;
- maintains correspondent accounts (IA **cb**),
- holds required reserves;
- maintains the correspondent account of State (IA **g**);
- has own capital.

In the model, "Central Bank" is an individual agent, and its behavior is described by **scenario**.

Start the description of CB with its operations with the currency exchange. **The dynamics of the exchange rate** (per dollar), we specify by a scenario. At this rate CB buys currency $V_{CB}(t)$ (sells if $V_{CB}(t) < 0$). It spends on purchases of US dollars $wV_{CB}(t) = w_w(t)\, V_{CB}(t)$, and stores the purchased currency in foreign exchange reserves $R_{CB}(t)$:

$$V_{CB}(t) = \frac{d}{dt}\, R_{CB}(t).$$

According to reported balance of CB [34], the reserves $R_{CB}(t)$ are the main and most of the time almost the only *asset* of CB.

In developed countries short-term loans to banks is the main asset of the Central Bank. But in Russia all the time except for the fall of 2008, banks have been investing in CB, instead of borrowing from CB. So in the model, we consider the net deposits (balance of deposits and loans) of Bank in Central Bank (see (6.26)) $Lc_{CB}(t)$:

$$dLc_{cb}(t) = \frac{d}{dt}\, Lc_{CB}(t).$$

Central Bank sets interest rate for loans $rl_{cb}(t)$ and deposits $rs_{cb}(t)$ (see (6.26)), and issues/deposits any amount Bank wants. For this reason the net inflow of deposits $dLc_{cb}(t)$ and interest payments on $rLc_{cb}(t)$ are considered as informational variables for the Central Bank (IA **cb**). Accordingly, the flow of payments on lending operations amounts to

$$KdLc(t) = dLc_{cb}(t) - rLc_{cb}(t).$$

In addition to these interest-bearing deposits, Bank keeps liquid money in Central Bank on correspondent accounts (including required reserves) and foreign currency accounts. Combined they form liabilities of CB $Liq_{cb}(t)$ (see (6.53)). Reserve rate $\zeta_{CB}(t)$ (see (6.27)) is set by scenario. Treasury bills $Ng_G(t)$ is another component of the liabilities of the Central Bank (see (6.53)).

Since all noncash payments, including payments to CB, are conducted through correspondent accounts or accounts of the Treasury, the sum of these accounts can only be changed by cashing of funds, for instance as a result of purchase of currency by Bank.

[30]Financing the budget by Central Bank is prohibited.

Table 6. The Central Bank planned variables.

Name	Dimension	Meaning
$Lc_{CB}(t)$	Money	Net loans from Bank to Central Bank
$Liq_{CB}(t)$	Money	Bank's liquidity on correspondent account within Central Bank
$Ng_{CB}(t)$	Money	Residuals of correspondent account of State within Central Bank
$R_{CB}(t)$	Currency	Currency reserves in dollars,
$V_{CB}(t)$	Currency/time	Flow of purchases/sales of currency
$rLc_{CB}(t)$	Money/time	Interest payments for loans from CB
$M0_{CB}(t)$	Cash	Cash in the economy

Central Bank makes money emission by this value, increasing the amount of money issued in circulation, $M0_{CB}(t)$:

$$\frac{d}{dt}\, M0_{CB}(t) = -\frac{d}{dt}\, Lc_{CB}(t) + rLc_{CB}(t) - \frac{d}{dt}\, Liq_{CB}(t) - \frac{d}{dt}\, Ng_{CB}(t) + w_w(t)\frac{d}{dt}\, R_{CB}(t). \tag{6.55}$$

We get the balance of Central Bank by integrating Eq. (6.55) over time, in the form

$$w_w(t)\, R_{CB}(t) = Lc_{CB}(t) + Ng_{CB}(t) + Liq_{CB}(t) + \Omega_{CB}(t),$$
$$\frac{d}{dt}\, \Omega_{CB}(t) = rLc_{CB}(t) - R_{CB}(t)\, \frac{d}{dt}\, w_w(t),$$

where $\Omega_{CB}(t)$ is one's own capital, and $-R_{CB}(t)\, \frac{d}{dt}\, w_w(t)$ is the gain from reevaluation of reserves, which Central Bank receives from devaluation of rouble.[31]

6.9. Individual Agent Importer IM. In the model, Importer:

- purchases imported product $Imp_{IM}(t)$ at the price $pw_{IM}(t)$ for the currency
$$wI_{IM}(t) = pw_{IM}(t)\, Imp(t);$$
- sells import to Trader (IA **i**) at the price of $p_i(t)$, yielding the rouble revenue
$$pI_{IM}(t) = p_i(t)\, Imp(t);$$
- pays out of this revenue import duties
$$Td_{IM}(t) = ni_x(t)\, p_i(t)\, Imp_{IM}(t)$$
at the rate of $ni_x(t)$ (see Fig. 6.7.3) and import tax at the rate of ti (see Sec. 6.7.2):
$$Ti_{IM}(t) = ti\, p_i(t)\, Imp_{IM}(t)$$
(IA **x**);
- buys currency $wI_{IM}(t)$ at the exchange (IA **w**), spending money
$$pwI_{IM}(t) = wI_{IM}(t)\, w_w(t).$$

In the model all value added is created by Producer and Bank. Hence the profit of artificial agent Importer, as well as the profit of Trader, is zero:

$$0 = pI_{IM}(t) - pwI_{IM}(t) - Td_{IM}(t) - Ti_{IM}(t).$$

[31] In principle profit of CB should be transferred to the budget, but as statistics show [34], a significant gain is obtained only from deevaluation, and it, in order to avoid inflation, is not being transferred into the budget.

Table 7. The Importer planned variables.

Name	Dimension	Meaning
$Imp_{IM}(t)$	Product/time	Import purchased by the importer from the external economy
$Td_{IM}(t)$	Money/time	Payment of duties
$Ti_{IM}(t)$	Money/time	Tax payments
$pI_{IM}(t)$	Money/time	Revenue from sales of imported product on the domestic market
$pw_{IM}(t)$	Currency/product	Price of imported product
$pwI_{IM}(t)$	Money/time	Expenditures for purchased currency for roubles on domestic market
$wI_{IM}(t)$	Currency/time	Costs in foreign currency of purchase of imported product in the external economy

From here we can get the relation between the external $pw_{IM}(t)$ and the internal $p_i(t)$ import prices:

$$p_i(t) = -\frac{pw_{IM}(t)\, w_w(t)}{-1 + ni_x(t) + ti}. \tag{6.56}$$

By the logic of the model the price $pw_{IM}(t)$ should be specified exogenously and $p_i(t)$ is found from (6.56). However, the value $pw_{IM}(t)$ is only modelled, whereas the price $p_i(t)$ is observable (see Fig. 4.1.1). Therefore, we first define $pw_{IM}(t)$ from (6.56) in terms of $p_i(t)$, and then extrapolate it together with $ni_x(t)$ over the forecast period and then use it as an exogenous variable.

6.10. Individual Agent Exporter EX. Exporter, just as Importer, plays in the model an artificial role of pure intermediary. However, in contrast to Importer it has two "additional" functions: it exports products and transfers foreign investments into the country. Export is described analogously to import (Sec. 6.9). Exporter

- buys export product $E_{EX}(t)$ from Trader at the price $pq_e(t)$ (IA **e**) for roubles in the amount of $pE_{EX}(t) = pq_e(t)\, E_{EX}(t)$;
- out of this sum it receives a VAT recovery (refund) (6.54) and pays export duties (see Fig. 6.7.3), that adds up to a net flow of payments into the budget

$$Tax_{EX}(t) = pE_{EX}(t)(ne_x(t) - te)\, pE_{EX}(t);$$

- sells the product $E_{EX}(t)$ on the foreign market at the price $pw_{EX}(t)$ and receives revenues in foreign currency

$$pwE_{EX}(t) = pw_{EX}(t)\, E_{EX}(t);$$

- sells the currency at the foreign exchange and receives revenue from this operation in roubles

$$w_w(t)\, pw_{EX}(t)\, E_{EX}(t);$$

- being a pure intermediary, does not receive any profit in the model; hence

$$0 = w_w(t)\, pw_{EX}(t)\, E_{EX}(t) - Tax_{EX}(t) - pE_{EX}(t).$$

The latter equation links the interior $pq_e(t)$ to the exterior $pw_{EX}(t)$ price of the export product:

$$pq_e(t) = pw_{EX}(t)\big(1 + te - ne_x(t)\big),$$

which is used to determine the unknown exterior price through the known interior price the same way as above.

Exporter in the model has the second role of a nonprofit mediator between foreign investors and Russian Bank. Formally, Exporter is assumed to be a holder of foreign deposits (in Russian roubles)

Table 8. The Exporter planned variables.

Name	Dimension	Meaning
$E_{EX}(t)$	Product/time	The volume of export product
$S_{EX}(t)$	Money	Foreign deposits in roubles
$Tax_{EX}(t)$	Money/time	Aggregate tax payments to the State budget
$V_{EX}(t)$	Currency/time	Currency inflow into the country in foreign deposits
$pExp_{EX}(t)$	Money/time	The revenues from sales of export product in roubles

$S_{EX}(t)$ in the Bank. The dynamics of deposits is described just like the Household's deposits (see Sec. 6.4). The variables of Exporter are described in Table 8.

It is assumed that Exporter receives a flow of currency deposit replenishment $VF_{EX}(t)$ from non-residents, exchanges it into Russian roubles at the currency market and puts to the account $S_{EX}(t)$, receives withdrawals $HF_{EX}(t) = \beta_s(t)\, S_{EX}(t)$ from Bank and interest payments $rSF_{EX}(t) = r_s(t)\, S_{EX}(t)$, exchanges them for currency and transfers outside the Russian border. In the model the deposits of non-residents $S_{EX}(t)$ ([34], Fig. 6.10.1) are known from official statistics and are considered to be exogenous.

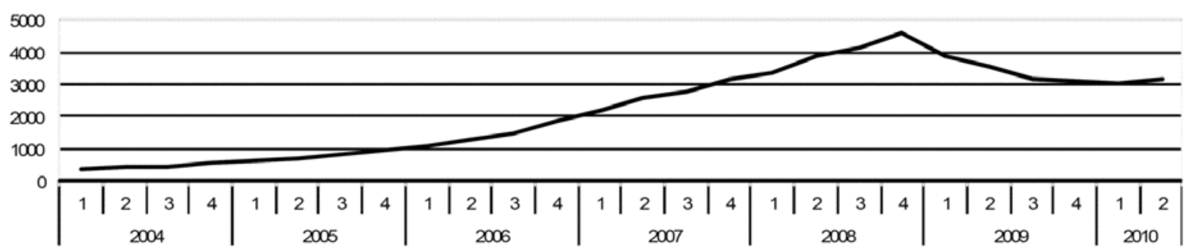

Fig. 6.10.1. Foreign deposits in Russian banks.

The dynamics equation for $S_{EX}(t)$ (after exclusion of auxiliary variables) is used to determine the unobserved amount of money transaction by foreigners across Russian border:

$$V_{EX}(t) = \frac{\frac{d}{dt} S_{EX}(t) - r_s(t)\, S_{EX}(t)}{w_w(t)}.$$

6.11. Interactions of Agents.

6.11.1. Variables and notation. Connections of agents' blocks are described by the channels of information transmission and by the equilibrium conditions of particular markets. For this, no new variables are used, but one may meet some auxiliary (redundant) variables that were used and excluded previously. The most important variables are assigned names without lower indexes. **The exogenous variables (only) receive names with affix "$\_s$" instead of lower index**.

6.11.2. The consumption product market c. In this market (IA) the supply of consumption product by Trader $C_Q(t)$ is brought into equilibrium with the demand of Household $C_H(t)$ and State $C_G(t)$:

$$0 = C_Q(t) - C_H(t) - G_c(t), \quad 0 = G_c(t) - G_G(t). \tag{6.57}$$

The flows of money payments in the opposite direction are also in equilibrium

$$0 = pG\_p_G(t) + pC_H(t) - pCH_Q(t) - pG\_p_Q(t).$$

The components of national accounts balance and price indexes are assigned names without indexes:

$$C_H(t) = C(t), \quad G_c(t) = G\_s(t), \quad pG\_p_G(t) = pGp\_s(t), \quad G\_p_c(t) = Gp\_s(t).$$

At the same time the actual payments $pGp\_s(t)$ for paid-up product $Gp\_s(t)$ and the volume of public consumption $G\_s(t)$ are exogenous variables, as well as all other State's budget expenditure series.

6.11.3. The investment product market j. The supply of the investment product by Trader $J_Q(t)$ and the demand for investment product by Producer $J_J(t)$ and State $J_G(t)$ are in equilibrium:

$$0 = J_Q(t) - J_J(t) - J_G(t), \tag{6.58}$$

The inverse-going payments are also in equilibrium

$$0 = pJ_J(t) + pJ_G(t) - pJ_Q(t),$$

Some variables are renamed

$$J_J(t) = J(t), \quad pJ_G(t) = pJG\_s(t), \quad J_G(t) = JG\_s(t)$$

and a new variable for public investments is introduced

$$J_j(t) = \frac{pJG\_s(t)}{p_j(t)}.$$

The payments $pJG\_s(t)$ for investment product $JG\_s(t)$ are exogenous variables, as well as all other budget expenditure variables.

6.11.4. The real GDP market y. On this market the supply of real GDP by Producer $Y_J(t)$ and the demand for real GDP by Trader $Y_Q(t)$ are in equilibrium. As well as the inverse-going payments:

$$0 = Y_J(t) - Y_Q(t), \quad 0 = pY_Q(t) - pY_J(t)$$

and the following variables are renamed

$$Y_J(t) = Y(t), \quad M_J(t) = M(t).$$

6.11.5. The interior product market u. At this market, Trader (as a mass agent) sells the interior product to himself, that is, his supply of the interior product $X_Q(t)$ is equilibrated by his demand for the interior product (inverse-directed payments get equilibrated as well):

$$0 = X_Q(t) - Xj_Q(t) - Xc_Q(t), \quad 0 = -pXI_Q(t) + pXE_Q(t).$$

One variable is renamed

$$X_Q(t) = X(t).$$

6.11.6. The import product market i. This market brings into equilibrium the import product supply by Importer $Imp_{IM}(t)$ and the demand by Trader $Ic_Q(t)$ and $Ij_Q(t)$, and counter payments as well:

$$0 = Imp_{IM}(t) - Ij_Q(t) - Ic_Q(t), \quad 0 = -pI_{IM}(t) + pI_Q(t).$$

The following variables are renamed

$$Imp_{IM}(t) = Ip(t), \quad pw_{IM} = pim\_s(t),$$

where $pim\_s(t)$ is the exogenously determined exterior price for the import product.

6.11.7. The export product market e. The export product supply by Trader $E_Q(t)$, demand for export product from Exporter $E_{EX}(t)$, and the counter payments are in equilibrium:

$$0 = E_Q(t) - E_{EX}(t), \quad 0 = pE_{EX}(t) - pE_Q(t).$$

The following variables are renamed

$$E_Q(t) = Ep(t), \quad pw_{EX}(t) = pex\_s(t).$$

6.11.8. The loan market l. The demand for bank loans from Producer $L_J(t)$ and the supply of bank loans by Bank $L_B(t)$ are equilibrated. Interest payment flows are also in equilibrium. The equilibrium condition on the loans market, including discarded loans, is:

$$0 = HL_B(t) - HL_J(t), \quad 0 = -K_B(t) + K_J(t), \quad 0 = ML_B(t) - ML_J(t).$$

Once Producer and Bank come to terms about loans, Producer orders the transfer of the interest payments from his current account in Bank and the accrual of newly acquired loans (loan extension):

$$0 = -rL_B(t) + rL_J(t), \quad 0 = -WdL_J(t) + WdL_B(t).$$

The variables are renamed

$$L_B(t) = L(t), \quad r_l(t) = r(t), \quad \beta_l(t) = bl\_s(t), \quad K_B(t) = K(t),$$

where $bl\_s(t)$ is exogenous parameter of loan duration.

6.11.9. The deposit market s. The demand for deposits by Bank $S_B(t)$ and the supply of deposits by Household $S_H(t)$ and nonresidents $S_{EX}(t)$ are brought into equilibrium. The equilibrium is represented in flows:

$$0 = -HV_B(t) + HF_{EX}(t) + H_H(t), \quad 0 = V_B(t) - V_H(t) - VF_{EX}(t).$$

When the clients replenish deposits, they bring money to Bank and receive interest payments:

$$0 = -WdS_B(t) + WdS_H(t) + WdS_{EX}(t), \quad 0 = rS_B(t) - rS_H(t) - rS_{EX}(t).$$

The following variables get renamed

$$S_H(t) = S(t), \quad S_{EX}(t) = Df\_s(t), \quad \beta_s(t) = bs\_s(t), \quad V_B(t) = V(t),$$

where $Df\_s(t)$ is the exogenous inflow of foreign deposits (see Fig. 6.10.1), and $bs\_s(t)$ is the exogenous parameter of deposit duration.

6.11.10. The interaction of Bank and Central Bank cb. In this interaction the supply of loans by Bank to Central Bank (CB) $Lc_B(t)$ is equilibrated by the demand for loans from Bank by CB $Lc_{CB}(t)$:

$$0 = -dLc_B(t) + dLc_{CB}(t), \quad 0 = -KdLc_{CB}(t) + KLc_B(t).$$

Also in this interaction the supply of rouble liquidity by Bank $A_B(t)$ (the residuals of current account within CB) is equilibrated by the demand for it by CB $Liq_{CB}(t)$:

$$0 = -dA_B(t) + dLiq_{cb}(t), \quad 0 = -KdLiq_{CB}(t) + KdA_B(t).$$

The following variables are renamed

$$\zeta_{cb}(t) = u\_s(t), \quad rl_{cb}(t) = rlcb\_s(t), \quad rs_{cb}(t) = rscb\_s(t), \quad Lc_B(t) = Lc(t),$$

where $u\_s(t)$ is the exogenously given reserve rate, $rlcb\_s(t)$ and $rscb\_s(t)$ are exogenous loan and deposit interest rates.

6.11.11. The interaction of State and CB g. In this block the supply of rouble residuals of the current account in CB by State $Ng_G(t)$ is equilibrated by the demand for it by CB $Ng_{CB}(t)$:

$$0 = dNg_g(t) - dNg_G(t), \quad 0 = -KdNg_{CB}(t) + KdNg_G(t).$$

The following variables are renamed

$$Ng_G(t) = Ng(t).$$

6.11.12. The labor market r. A satisfactory description for this market in the case of Russia is still in progress. Partially it can be explained by low development of the labor market, since wages significantly differ across regions and industries; partially due to lack of reliable data on actual wages paid. We can no more than note that the commonly acknowledged way to model a labor market by means of "labor aversion" and "human capital" is not applicable to the Russian economy.

Hence, currently the way out is only to consider labor as an exogenous variable $R\_s(t)$. In Sec. 6.1 above we have already assumed labor as an informational variable $R_r(t)$. Then all we have to do is to set equal the demand for labor by Producer and the labor supply by Household:

$$0 = -R_J(t) + R_r(t), \quad R_r(t) = R_J(t).$$

Wages are counter payments:

$$0 = -sR_H(t) + sR_J(t).$$

The following variables are renamed

$$R_r(t) = R\_s(t), \quad s_r(t) = s(t).$$

6.11.13. Taxes, duties, and subsidies x. The interaction describes tax transfers to the public budget:

$$0 = BudInc_x(t) - Tax_J(t) - Tax_{EX}(t) - Tax_{IM}(t)$$

and payments of scenario subsidies and transfers from the budget:

$$0 = Tr_G(t) - SB_x(t), \quad 0 = SubPr_G(t) - Sub_x(t).$$

Though public consumption and investments are budget expenditures they are not described in this block. Public consumption is outlined in Sec. 6.11.2 and public investments are in Sec. 6.11.3.

The following variables are renamed

$$SubPr_G(t) = SP\_s(t), \quad Tr_G(t) = Su\_s(t), \quad ne_x(t) = e\_s(t),$$
$$ni_x(t) = i\_s(t), \quad BudInc_x(t) = BudInc(t),$$

where $SP\_s(t)$, $Su\_s(t)$, $e\_s(t)$, and $i\_s(t)$ are exogenous subsidies to Producer, transfers to Household, and the rates of export and import duties.

6.11.14. Producer's capital control up. The current and the subsequent two blocks describe property relations. Such approach distinguishes the intertemporal equilibrium models with control of capital (IEMCC).

Owner receives information about rates and coefficients of Producer's capital control equation, which are calculated in the Producer's problem (see Sec. 6.1):

$$\theta_{up} = \theta_J, \quad \nu_{up}(t) = \nu_J(t), \quad \rho_{up}(t) = \rho_J(t), \quad f_{up}(t) = f_J(t). \tag{6.59}$$

Producer learns from Owner about dividend rate:

$$Ub_{up}(t) = zp_C(t),$$

Producer's dividends are transferred to Owner:

$$0 = -Zp_C(t) + Z_J(t).$$

6.11.15. Bank's capital control ub. In form and meaning this interaction is highly similar to the previous one but for variables indexation:

$$\theta_{ub} = \theta_B, \quad \nu_{ub}(t) = \nu_B(t), \quad \rho_{ub}(t) = \rho_B(t), \quad Ub_{ub}(t) = zb_C(t),$$
$$f_{ub}(t) = f_B(t), \quad 0 = -Zb_C(t) + Z_B(t).$$

6.11.16. Trade dividends uq. Since Trader has no capital at all this interaction has no capital control. Owner receives flow of dividends from Trader $Z_Q(t)$ which are Owner's informational variable $Zq_{uq}(t)$. Equilibrating these flows gives:

$$0 = -Zq_{uq}(t) + Z_Q(t).$$

6.11.17. The currency market w. The demand for foreign currency (primarily US dollars) of Owner $V_C(t)$, Importer $wI_{IM}(t)$, Central Bank $V_{CB}(t)$, and Bank $dQ_B(t)$ as well as the supply of foreign currency by Exporter $pwE_{EX}(t)$ and nonresident depositors $V_{EX}(t)$ are equilibrated:

$$0 = V_{EX}(t) + pwE_{EX}(t) - V_C(t) - V_{CB}(t) - wI_{IM}(t) - dQ_B(t).$$

Corresponding rouble payments are also equilibrated:

$$0 = -wV_{EX}(t) + wV_C(t) + wV_{CB}(t) - pExp_{EX}(t) + pwI_{IM}(t) + WdQ_B(t).$$

Some variables are renamed

$$w_w(t) = w\_s(t), \quad R_{CB}(t) = ZVR(t),$$

where $w\_s(t)$ is exogenous USD exchange rate in roubles.

6.11.18. Cashless payments n. Producer and Bank see changes in current accounts the same way:

$$0 = -NdN_J(t) + dN_B(t)$$

The corresponding flows of current accounts take place:

$$0 = -WdN_B(t) + KdN_J(t).$$

The expectations of Bank for current accounts are fulfilled according to the Rational Expectation principle:

$$N_n(t) = N_J(t).$$

The renamed variables are

$$N_J(t) = N(t), \quad M0_{CB}(t) = M0(t), \quad W_B(t) = W(t).$$

7. The Calculation Results of the Russian Economy Model

7.1. The Final Model Description. Formally, the model consists of the equations from the Secs. 6.2.3, 6.3.3, 6.4.2, 6.5.3, 6.6, 6.7.1, 6.8–6.11, including also the equations for the auxiliary variables which were omitted in agents' descriptions. Altogether we obtain a system of 175 equations. By means of ECOMOD correctness of the system of balances was checked (see Sec. 5.2). Since the balances for each financial instrument are complete we derive first integrals for the system of equations. The unknown constants θ_J, θ_B (see Secs. 6.11.14, 6.11.15) can be set equal to one by adjusting the initial values of unobserved variables $Kp_C(t0)$ and $Kb_C(t0)$ (6.48) and (6.58).

Almost half of the derived equations can be regarded as expressions for insignificant auxiliary variables, which are detected by ECOMOD system automatically. Then the system is manually simplified by means of specially customized MAPLE routines. At this stage the complementary slackness conditions are resolved (or regularized) according to Sec. 6.3.2. It is worth noting that once small changes in the original description of agents take place it is easy to replicate these operations automatically to obtain a new model. After all simplifications we obtain a model that consists of 38 equalities:

$$\begin{aligned} N(t) &= \big(p_y(t)\,Y(t) + mn3Df\_s(t)\big)\big(-mn1\rho_J(t) + mn2 + mn4\sin(1/2\pi t + mn5)\,o\_s(t)\big) \\ &\quad + \tau_s s(t)\,R\_s(t) - \tau_j\big(-J(t) + JG\_s(t)\big)\,p_j(t) + \tau_l L(t), \qquad (7.1) \\ K(t) &= \big(p_y(t)Y(t) + lfL(t) + lf2p_j(t)\,J(t) \\ &\quad + lf3s(t)R\_s(t) + lf4SP\_s(t)\big)\big(-mk1k(t) + mk2 + mk3r(t)\big), \\ e^{\frac{r_S(t)-\iota_H(t)}{ah0}} &= \left(\frac{A_H(t)}{AH0}\right)^{ah1}\left(\frac{p_c(t)C(t)}{pC0}\right)^{ah2}, \\ \big(ddj - ej\rho_J(t) &+ ej2Dw\_s(t)\big)\big(p_y(t)Y(t) + ejlL(t) + ejnN(t)\big) \\ &= \big(p_j(t) + \rho_J(t)\tau_j p_j(t)\big)\,M(t) + N(t) + (-k(t) - 1)L(t), \end{aligned}$$

$$
\begin{aligned}
&e^{-\left(-\rho_B(t)+\frac{\frac{d}{dt}w\_s(t)}{w\_s(t)}+db\right)eb^{-1}} P(t) = (E(t) - u\_s(t) - 1)\\
&\quad - \Bigg(-\sigma_r(t) + r(t) - \frac{(-m + chiB)\, r_s(t)}{mu\_s(t) - m + \tau bs\_s(t)}\\
&\quad - \frac{(-chiB + chiBu\_s(t) + \tau bs\_s(t))rlcb\_s(t)}{mu\_s(t) - m + \tau bs\_s(t)} - \rho_B(t) chiB\Bigg) L(t)\big(-\sigma - bl\_s(t) - \rho_B(t)\big)^{-1}\\
&\quad + Liq(t) + \big(S(t) + Df\_s(t)\big)u\_s(t) + u\_s(t)\,\big(P(t) - S(t) - Df\_s(t)\big),\\
&\frac{d}{dt}\, p_j(t) = \iota_J(t) p_j(t), \quad \frac{d}{dt} M(t) = J(t) - \kappa M(t),\\
&\frac{d}{dt}\rho_J(t) = -\frac{-\kappa + \iota_J(t)}{\tau_j} + \frac{p_y(t) A(ty-1)}{\tau_j p_j(t)} + \big(\rho_J(t)\big)^2 - \frac{-\kappa\tau_j - 1 + \iota_J(t)\tau_j)\rho_J(t)}{\tau_j},\\
&-\frac{d}{dt}k(t) = (bl\_s(t) + \sigma)(-k(t) - 1) + bl\_s(t) + r(t) + (\tau_l - k(t) - 1)\rho_J(t),\\
&Q_B(t) = \Bigg(aq - \frac{(\frac{d}{dt}w\_s(t))bq(u\_s(t)-1)}{w\_s(t) r_s(t)} + bq(u\_s(t) - 1)\\
&\quad \times \Bigg(\frac{-rlcb\_s(t)u\_s(t) + rlcb\_s(t) + rlcb\_s(t)mu\_s(t) - rlcb\_s(t)m}{mu\_s(t) - m + \tau bs\_s(t)}\\
&\quad + \frac{(-1+m)r_s(t)}{mu\_s(t) - m + \tau bs\_s(t)}\Bigg)\big(r_s(t)\big)^{-1}\Bigg)P(t),\\
&0 = -\frac{d}{dt}\, N(t) + SP\_s(t) - tes\, s(t)\, R\_s(t) - s(t)\, R\_s(t) - R\_s(t)\, Bp_y(t)(ty-1)\, e^{bt-bt0}\\
&\quad + \big(-J(t) + JG\_s(t)\big)\, p_j(t) - p_y(t)\, A(ty-1)M(t) + \big(-bl\_s(t) - r(t)\big)\, L(t)\\
&\quad + \big(-bs\_s(t) - r_s(t)\big)\big(S(t) + Df\_s(t)\big) + bs\_s(t)\big(S(t) + Df\_s(t)\big)\\
&\quad + rLc(t) + \big(-bl\_s(t) - r(t)\big)\, Lc(t) - \frac{d}{dt}Lc(t) - \frac{d}{dt}\, Liq(t)\\
&\quad + \Bigg(\big(bl\_s(t) + r(t)\big)E(t) + \big(-bl\_s(t) - r(t)\big)u\_s(t) - \frac{d}{dt}u\_s(t)\Bigg)P(t)\\
&\quad + (-u\_s(t) + 1)\frac{d}{dt}\, P(t) + \frac{Q_B(t)\frac{d}{dt}w\_s(t)}{w\_s(t)} - p_c(t)\big(G\_s(t) - Gp\_s(t)\big) - w\_s(t)\, V_C(t),\\
&0 = -\frac{d}{dt}\, Ng(t) + BudInc(t) - tj\, p_j(t)\, JG\_s(t) - p_c(t)\, Gp\_s(t) - p_j(t)\, JG\_s(t) - SP\_s(t) - Su\_s(t),\\
&\frac{d}{dt}\, Q_H(t) = C(t) - \mu Q_H(t), \quad \frac{d}{dt}\, p_c(t) = \iota_H(t) p_c(t), \quad 0 = r_s(t) + \mu - \iota_H(t) - \big(q(t)\big)^{-1},\\
&\frac{d}{dt}q(t) = -1 + \left(\mu + \Delta + \frac{\eta C(t)}{Q_H(t)} - \eta\mu\right)q(t),\\
&\frac{d}{dt}\, M0(t) = -\frac{d}{dt}\, Lc(t) + rLc(t) - \frac{d}{dt}\, Liq_{CB}(t) - \frac{d}{dt}\, Ng(t) + w\_s(t)\frac{d}{dt}\, ZVR(t),\\
&\frac{d}{dt}ZVR(t) = -V_C(t) - pim\_s(t)\, Ip(t) + pex\_s(t) Ep(t)\\
&\quad + \frac{-r_s(t)Df\_s(t) - \frac{d}{dt}Q_B(t)}{w\_s(t)} + \frac{Q_B(t)\frac{d}{dt}w\_s(t)}{(w\_s(t))^2} + \frac{\frac{d}{dt}Df\_s(t)}{w\_s(t)},\\
&\frac{d}{dt}L(t) = (-bl\_s(t) - \sigma)\, L(t) + K(t), \quad s(t) = -\frac{Bp_y(t)\, e^{b(t-t0)}(ty-1)}{tes + 1 + \rho_J(t)\, \tau_s},
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
rLc(t) &= \Big(-\frac{-rlcb\_s(t)brl1\,rlcb\_s(t)\,u\_s(t)}{-brl1+brl1u\_s(t)-brl2}+\frac{rscb\_s(t)brl2}{-brl1+brl1u\_s(t)-brl2}\Big)L(t)+\\
&+\Big(\frac{brl1rlcb\_s(t)u\_s(t)}{-brl1+brl1u\_s(t)-brl2}+\frac{escb\_s(t)u\_s(t)brl2}{-brl1+brl1u\_s(t)-brl2}-\frac{brl1rlcb\_s(t)(u\_s(t))^2}{-brl1+brl1u\_s(t)-brl2}\\
&-\frac{brl1u\_s(t)arl2}{-brl1+brl1u\_s(t)-brl2}+\frac{brl1arl2}{-brl1+brl1u\_s(t)-brl2}-\frac{brl1brl2}{-brl1+brl1u\_s(t)-brl2}\\
&+\frac{E(t)brl1rlcb\_s(t)u\_s(t)}{-brl1+brl1u\_s(t)-brl2}-\frac{E(t)rlcb\_s(t)brl1}{-brl1+brl1u\_s(t)-brl2}+\frac{brl1u\_s(t)brl2}{-brl1+brl1u\_s(t)-brl2}\\
&-\frac{E(t)rscb\_s(t)brl2}{-brl1+brl1u\_s(t)-brl2}+\frac{arl1brl2}{-brl1+brl1u\_s(t)-brl2}\Big)P(t),\\
\rho_B(t) &= \frac{m+aliq-1+mu\_s(t)-arho}{(-1+m)brho}\\
&-(brho+bliq)\Big(\frac{-rlcb\_s(t)u\_s(t)+rlcb\_s(t)+rlcb\_s(t)mu\_s(t)-rlcb\_s(t)m+(-1+m)r_s(t)}{mu\_s(t)-m+\tau bs\_s(t)}\Big)\\
&\times(-1+m)^{-1}brho^{-1}-\frac{E(t)}{brho}\\
&+\Big(-\frac{(-m+chiB)L(t)}{(-1+m)brho}+\frac{(\tau bs\_s(t)-m)S(t)\,Df\_s(t))}{(-1+m)brho}-\frac{Q_B(t)}{brho}\Big)\big(P(t)\big)^{-1},\\
Liq(t) &= mL(t)+(\tau bs\_s(t)-m)\big(S(t)+Df\_s(t)\big)+(1-m)Q_B(t)+\Big(aliq-mE(t)+mu\_s(t)+m\\
&-bliq\Big(\frac{-rlcb\_s(t)u\_s(t)+rlcb\_s(t)+rlcb\_s(t)mu\_s(t)-rlcb\_s(t)m+(-1+m)r_s(t)}{mu\_s(t)-m+\tau bs\_s(t)}\Big)\Big)P(t),\\
E(t) &= e1+e2+e3u\_s(t)+e3\\
&+\Big(-\frac{(e3+e1\,m)Liq(t)}{m}+(chiBe1+e3)\,L(t)+\frac{e3(\tau bs\_s(t)-m)(S(t)+Df\_s(t))}{m}\Big)\big(P(t)\big)^{-1},\\
K(t) &= \Big(\Big(\Big(-\frac{\rho_B(t)\,chiB}{bl\_s(t)+\rho_B(t)}+\frac{r_s(t)}{(u\_s(t)-1)(bl\_s(t)+\rho_B(t))}+\frac{r(t)}{bl\_s(t)+\rho_B(t)}\\
&-(-chiB+chiBu\_s(t)+\tau bs\_s(t))\Big(\frac{-rlcb\_s(t)u\_s(t)+rlcb\_s(t)+rlcb\_s(t)mu\_s(t)-rlcb\_s(t)\,m}{mu\_s(t)-m+\tau bs\_s(t)}\\
&+\frac{(-1+m)\,r_s(t)}{mu\_s(t)-m+\tau bs\_s(t)}\Big)\big(bl\_s(t)+\rho_B(t)\big)^{-1}(-1+m)^{-1}(u\_s(t)-1)^{-1}\Big)bk+ak\Big)P(t),\\
0 &= -BudInc(t)+ty\cdot p_y(t)\big(A\cdot M(t)+Be^{b(t-t0)}R\_s(t)\big)+tes\cdot s(t)R\_s(t)\\
&+w\_s(t)Ep(t)(e\_s(t)+te)\,pex\_s(t)-\frac{pim\_s(t)Ip(t)w\_s(t)(i\_s(t)+ti)}{-1+i\_s(t)+ti}+tj\,p_j(t)JG\_s(t),\\
Lc(t) &= \big(E(t)-u\_s(t)\big)P(t)-L(t),\quad Liq_{CB}(t)=Liq(t)-Q_B(t)+P(t)\,u\_s(t),\\
N(t) &= P(t)-S(t)-Df\_s(t),\quad Y(t)=A\cdot M(t)+B\cdot e^{b(t-t0)}R\_s(t),\\
Ip(t) &= D_1(qc)\big(p_i(t),p_u(t)\big)\big(C(t)+G\_s(t)\big)+D_1(qj)\big(p_i(t),p_u(t)\big)\,J(t),\\
Ep(t) &= D_2(qy)\big(p_u(t),pq_e(t)\big)Y(t),\quad p_c(t)=qc\big(p_i(t),p_u(t)\big),\quad p_j(t)=qj\big(p_i(t),p_u(t)\big),\\
p_y(t) &= qy\big(p_u(t),pq_e(t)\big),\quad p_i(t)=-\frac{pim\_s(t)w\_s(t)}{-1+i\_s(t)+ti},\quad pq_e(t)=-w\_s(t)(e\_s(t)+te-1)\,pex\_s(t),\\
D_1(qy)&\big(p_u(t),pq_e(t)\big)Y(t)-D_2(qj)\big(p_i(t),p_u(t)\big)J(t)=D_2(qc)\big(p_i(t),p_u(t)\big)\big(C(t)+G\_s(t)\big). \qquad (7.3)
\end{aligned}
$$

The system consists of 38 equalities for 38 unknown variables, out of them 18 equalities are dynamic. The variables with names that have suffix $\_s$ are exogenous. Their description is given in Table 9.

Table 9. The exogenous variables.

Name	Dimension	Meaning
Non-controlled exterior factors		
$R\_s(t)$	Labor/time	The quantity of employees, e.g., public sector
$bl\_s(t)$	1/time	Loan duration
$bs\_s(t)$	1/time	Deposit duration
$pim\_s(t)$	Currency/product	Exterior (US dollar) import price
$pex\_s(t)$	Currency/product	Exterior (US dollar) export price
$Df\_s(t)$	Money	Deposits of nonresidents
Public policy factors		
$G\_s(t)$	Product/time	Real public consumption
$e\_s(t)$	1	Export duty rate
$i\_s(t)$	1	Import duty rate
$u\_s(t)$	1	Central Bank reserve rate
$w\_s(t)$	Money/currency	US dollar exchange rate (in roubles)
$Gp\_s(t)$	Product/time	Paid real public consumption
$Dw\_s(t)$	1/time	US dollar exchange rate growth rate
$JG\_s(t)$	Product/time	Real public investments
$SP\_s(t)$	Money/time	Subsidies to Producer
$Su\_s(t)$	Money/time	Transfers to Household
$rlcb\_s(t)$	1/time	Bank to Central Bank loan interest rate
$rscb\_s(t)$	1/time	Central Bank to Bank deposit interest rate

Boundary values of phase variables ($Lc(t)$, $L(t)$, $M0(t)$, $M(t)$, $Ng(t)$, $N(t)$, $P(t)$, $Q_H(t)$, $Liq_{CB}(t)$, $Liq(t)$, and $ZVR(t)$) at $t = t0$ and $t = T$ are utilized as adjusting parameters, bound by the capital growth conditions (6.18), (6.35), (6.40), and (6.50).

On the whole, the model has 70 constant parameters, including 40 parameters known from statistical data, with the other 30 used for model identification. As an initial solution for the boundary value problem we take the auto-model solution derived under an assumption that all exogenous variables grow exponentially with time.

7.2. The Calculation and Identification Algorithm.

7.2.1. Auto-model solutions and balanced growth. Scale invariance is typical of dynamic models of economy. Hence it is reasonable to start the analysis with the study of solutions which are auto-model with respect to homothety. For example, the well known solutions of **balanced growth** in which all additive (extensive) values grow with the same constant growth rate. Most results in economic theory are obtained by comparing the regimes of balanced growth in simple abstract economic models.

However a model can allow for a more diverse group of homothety which corresponds to solutions with different growth rates for different variables. For instance, the gap between financial and material growth rates refers to inflation, and the spread in growth rates between production and resource costs refers to technological growth rate. We call the solutions with such properties **auto-model** solutions. Auto-model solutions serve as a starting point for model calculation and identification.

Consider the system (7.1)–(7.3) in a vector form:

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{y}(t), \mathbf{s}(t), \mathbf{p}), \quad 0 = \mathbf{G}(\mathbf{x}(t), \mathbf{y}(t), \mathbf{s}(t), \mathbf{p}), \tag{7.4}$$

$$0 = \mathbf{B}(\mathbf{x}(t0), \mathbf{y}(t0), \mathbf{s}(t0), \mathbf{x}(T), \mathbf{y}(T), \mathbf{s}(T), \mathbf{p}), \tag{7.5}$$

where $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are lists of variables, $\mathbf{s}(t)$ is a list of exogenous variables, and $\mathbf{p}$ is a list of constant parameters. In order to find auto-model solutions of the system (7.4), (7.5), that correspond to homothety group, we substitute

$$\mathbf{x}_i(t) = \bar{\mathbf{x}}_i e^{\boldsymbol{\xi}_j(t-t0)}, \quad \mathbf{y}_j(t) = \bar{\mathbf{y}}_j e^{\boldsymbol{\eta}_j(t-t0)}, \quad \mathbf{s}_k(t) = \bar{\mathbf{s}}_k e^{\boldsymbol{\zeta}_k(t-t0)} \tag{7.6}$$

and find the constants $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{s}}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$, under which the system (7.4), (7.5) will be satisfied for all t.

Solving such a system head-on in order to find the growth rates of 20 variables is almost impossible even for computer algebra systems. Besides, in our case, such an approach faces not only technical but also principal difficulties.

What was said about symmetry and auto-model solutions above refers to closed (autonomous) systems. The models of real economy, however, typically are not closed. In these models one can find analogues of auto-model solutions only if the exogenous variables $\mathbf{s}(t)$ are approximated by exponents. At the same time it is not obvious which values of exogenous variable growth rates should be chosen. We believe that we managed to find a rational way to solve this problem.

We propose that we look for auto-model solutions where values with the same dimension have the same growth rate. While checking the dimensions of the variables $\mathbf{x}_i(t)$, $\mathbf{y}_j(t)$, and $\mathbf{s}_k(t)$, a certain dimension (production of degrees of basic dimensions) was attributed to each variable. If in this expression the dimension of *time* is replaced by 1, and the basic dimension ω is replaced by the expression $e^{\lambda_\omega(t-t0)}$, where λ_ω is an indefinite coefficient (interpreted as "dimension growth rate"), then the expression of a variable dimension converted to an exponent. The growth rate of the latter is the growth rate of the corresponding variable provided that the variables of dimension have the same growth rate. Thus from expressions for dimensions we obtain an expression for the growth rates of the variables $\mathbf{x}(t)$, $\mathbf{y}(t)$, and $\mathbf{s}(t)$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\lambda}), \quad \boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{\lambda}), \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\lambda}), \tag{7.7}$$

where $\boldsymbol{\lambda}$ is a yet unknown vector of growth rates for basic dimensions, which we find in the form of (7.6).

This approach has the following advantages:

- the number of basic dimensions is less than the number of variables; hence the system of equations for auto-model solution parameters should be relatively simple;
- basic dimensions are attributed to the additive values of stocks of assets; hence the auto-model solutions should be directly analogous to traditional balanced growth;
- the dimensions of exogenous variables are determined at the stage of assembly in the ECOMOD system; hence their growth rates are determined based on growth rates of endogenous variables;
- solvability of the system with respect to rates $\boldsymbol{\lambda}$ and obtaining an algebraic system for amplitudes $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, and $\bar{\mathbf{s}}$ are guaranteed by the fact that the system of equations for $\boldsymbol{\lambda}$ is a subsystem of the equations for homothety group generators (allowed by the model), and solvability for extension group generators is a criterion for correctness of the system of dimensions;
- since dimensions for all variables and parameters are expressed in terms of basic dimensions, the assembly of a system for finding the auto-model solution can be automatized. It was realized in the ECOMOD system.

Broadly speaking, the proposed method does not allow us to find all solutions of the form (7.6). However, we deliberately confine the analysis to those solutions in which variables with the same dimension have the same growth rate, because such an implication corresponds to our view of economic growth. That is, we impose an additional requirement on the equations of the model, which is not listed in the model, but reflects a substantial sense: existence of growth rates that comply with the system of assets dimensions.

7.2.2. Determining the parameters from auto-model solutions. We used the procedure described above to find the auto-model solutions that are compatible with the basic dimensions of additive values. Application to the system of equations of the model (7.4), (7.5) results in expressions

$$0 = \Phi(\boldsymbol{\lambda}, \gamma, \mathbf{p}), \quad 0 = \boldsymbol{\Gamma}(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}}, \boldsymbol{\lambda}, \gamma, \mathbf{p}), \quad 0 < \mathbf{H}(\mathbf{p}) \tag{7.8}$$

for the amplitudes $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, and $\bar{\mathbf{s}}$ of the auto-model solution (7.6), (7.7) and of the vector of growth rates of basic dimensions of the model $\boldsymbol{\lambda}$.

The first equality in (7.8) is derived from the requirement that when (7.6) is substituted in (7.4) the exponents are cancelled out. The second equality is derived from the equalities (7.4) and (7.5) after cancellation of the exponents, and the last inequality $0 < \mathbf{H}(\mathbf{p})$ is imposed by default condition on parameters to be positive and other auxiliary conditions combined in groups "conditions on parameters" of the ECOMOD system.

The system (7.8) has fundamental importance for the solution. It is used to identify the values of parameters and to exclude the boundary conditions. We contemplate the system (7.8) as an underdetermined system of equations for the whole set of variables and the parameters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{s}$, $\boldsymbol{\lambda}$, γ, and $\mathbf{p}$. Then choosing a feasible solution point for the system $\bar{\mathbf{x}}^*$, $\bar{\mathbf{y}}^*$, $\bar{\mathbf{s}}^*$, $\boldsymbol{\lambda}^*$, γ^*, $\mathbf{p}^*$, we apply an auto-model solution

$$\mathbf{x}_i(t) = \bar{\mathbf{x}}_i^* \, e^{\xi_j(\boldsymbol{\lambda}^*)\cdot(t-t0)}, \quad \mathbf{y}_j(t) = \bar{\mathbf{y}}_j^* \, e^{\eta_j(\boldsymbol{\lambda}^*)\cdot(t-t0)}, \quad \mathbf{s}_k(t) = \bar{\mathbf{s}}_k^* \, e^{\zeta_k(\boldsymbol{\lambda}^*)\cdot(t-t0)} \tag{7.9}$$

as an initial approximation for solution of the system of equations for the parameters γ^* and $\mathbf{p}^*$.

In the method proposed above the choice of a point that is feasible for (7.8) $\bar{\mathbf{x}}^*$, $\bar{\mathbf{y}}^*$, $\bar{\mathbf{s}}^*$, $\boldsymbol{\lambda}^*$, γ^*, $\mathbf{p}^*$ corresponds to "fitting" the values of the parameters. This task arises whenever a mathematical model is used for applied calculations. If the model can be reduced to a dynamic system, then the solution either exists for any set of parameters, or if there is no solution at all, it can be seen from the very beginning. In our case the model is reduced to a boundary value problem; hence the existence of a solution is not so easy to see, and we have to identify the parameters together with an initial approximation to the solution.

7.2.3. Normalization and boundary conditions exclusion. An auto-model solution (7.9) satisfies the conditions (7.4) and (7.5), if the exogenous variables are described by $\mathbf{s}_k(t) = \bar{\mathbf{s}}_k^* e^{\zeta_k(\boldsymbol{\lambda}^*)\cdot(t-t0)}$. In reality, exogenous variables do not change in time exponentially, but in a much more complicated way. Hence it is necessary to be able to solve the general case. The idea we describe below is to search for a solution as a bounded correction term to the auto-model solution. To implement this idea we have to renormalize the variables on the chosen auto-model solution by making the following substitutions:

$$\mathbf{x}(t) = \tilde{\mathbf{x}}_i \, e^{\xi_j(\boldsymbol{\lambda}^*)\cdot(t-t0)}, \quad \mathbf{y}_j(t) = \tilde{\mathbf{y}}_j \, e^{\eta_j(\boldsymbol{\lambda}^*)\cdot(t-t0)}, \quad \mathbf{s}_k(t) = \tilde{\mathbf{s}}_k \, e^{\zeta_k(\boldsymbol{\lambda}^*)\cdot(t-t0)}.$$

Then Eqs. (7.4) turn into

$$\frac{d}{dt}\tilde{\mathbf{x}}(t) = \mathbf{F}^*\big(\tilde{\mathbf{x}}(t), \tilde{\mathbf{y}}(t), \tilde{\mathbf{s}}(t)\big), \quad 0 = \mathbf{G}^*\big(\tilde{\mathbf{x}}(t), \tilde{\mathbf{y}}(t), \tilde{\mathbf{s}}(t)\big), \tag{7.10}$$

where $\tilde{\mathbf{s}}(t)$ are exogenous. The superscript "*" means the parameters and growth rates of auto-model solution belong to the admissible set $\bar{\mathbf{x}}^*$, $\bar{\mathbf{y}}^*$, $\bar{\mathbf{s}}^*$, $\boldsymbol{\lambda}^*$, γ^*, $\mathbf{p}^*$.

The main difficulty is that the system (7.10) is highly unstable with respect to fluctuations of boundary conditions. The task remains correct if we are able to develop bounded solutions of the system. In general they are not necessarily unique. One could take advantage of such freedom and try to satisfy boundary conditions at $t = T$. However, there are grounds for believing that boundary terminal conditions are artificial from the standpoint of economic theory. So we propose to utilize the degrees of freedom in the choice of finite solutions to verify the model based on the official statistical data by adjusting model parameters. Note that such an approach does not ignore the initial boundary conditions. We define the finiteness of a solution based on an auto-model solution which satisfies the boundary value problem if we are provided an exponential approximation of the exogenous variables.

7.2.4. Switching to discrete time and exogenous variables homotopy. Commonly, statistical data are available in discrete units of time: month, quarter, year. In case the tick (the time difference between two consequent observation) is large, the difference scheme matters (i.e., the way derivatives are interpreted in discrete time). Based on the experience of previous models [7], we use an implicit scheme for approximation of the derivatives:

$$\frac{d}{dt}\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t-1).$$

After that, the system (7.10) converts into

$$\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t-1) + \mathbf{F}^*\big(\tilde{\mathbf{x}}(t), \tilde{\mathbf{y}}(t), \tilde{\mathbf{s}}(t)\big), \quad 0 = \mathbf{G}^*\big(\tilde{\mathbf{x}}(t), \tilde{\mathbf{y}}(t), \tilde{\mathbf{s}}(t)\big), \quad t = 1, 2, \ldots, T. \tag{7.11}$$

The task is to find solutions for the system (7.11) that differ from $\tilde{\mathbf{x}}^*$, $\tilde{\mathbf{y}}^*$ by not more than one degree. The proposed method guarantees achievement of such finite solutions and simultaneously identifies boundary values. By construction the pair $\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}^*$, $\tilde{\mathbf{y}}(t) = \tilde{\mathbf{y}}^*$ is a solution for the system (7.11) provided that $\tilde{\mathbf{s}}(t) = \tilde{\mathbf{s}}^*$. However, commonly it is hard to calculate the solution for the system (7.11) if we are provided certain exogenous variable $\tilde{\mathbf{s}}(t)$, assuming $\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}^*$, $\tilde{\mathbf{y}}(t) = \tilde{\mathbf{y}}^*$ as an initial solution, which is why we used the immersion method. The required values $\tilde{\mathbf{s}}(t)$ were connected with the "convenient" $\tilde{\mathbf{s}}^*$ curve in the space of sequences (by homotopy):

$$\mathbf{s}_u(t) = (1-u)\mathbf{s}^* + u\tilde{\mathbf{s}}(t).$$

Substituting $\mathbf{s}_u(t)$ into the system (7.11) instead of $\tilde{\mathbf{s}}(t)$, we obtain

$$\mathbf{x}_u(t) = \mathbf{x}_u(t-1) + \mathbf{F}^*\big(\mathbf{x}_u(t), \mathbf{y}_u(t), \mathbf{s}_u(t)\big), \quad 0 = \mathbf{G}^*\big(\mathbf{x}_u(t), \mathbf{y}_u(t), \mathbf{s}_u(t)\big), \quad t = t0, \ldots, T. \tag{7.12}$$

Obviously $\mathbf{x}_0(t) = \tilde{\mathbf{x}}^*$, $\mathbf{y}_0(t) = \tilde{\mathbf{y}}^*$ is a solution of the system under $u = 0$, and when $u = 1$, the values $\tilde{\mathbf{x}}(t) = \mathbf{x}_1(t)$, $\tilde{\mathbf{y}}(t) = \mathbf{y}_1(t)$ satisfy (7.11). The calculation method implies a gradual increase of the parameter u according to the following scheme:

- in the beginning, assume $u_0 = 0$. At this value of the homotopy parameter u, the solution $\mathbf{x}_{u_0}(t) = \tilde{\mathbf{x}}^*$, $\mathbf{y}_{u_0}(t) = \tilde{\mathbf{y}}^*$ is known;
- for the step $n \geq 0$ for $u_n < 1$ let there be a known solution $\mathbf{x}_{u_n}(t)$, $\mathbf{y}_{u_n}(t)$. Taking this solution as an initial approximation, try to solve the equations at $u = 1$. In case of success the task is solved. Otherwise try to find the solution atfor the values

  $$u = \frac{1}{2} + \frac{u_n}{2}, \frac{1}{4} + \frac{3u_n}{4}, \ldots,$$

  approaching u_n, every step using $\mathbf{x}_{u_n}(t)$, $\mathbf{y}_{u_n}(t)$ as an initial solution. The first value of u for which the solution is obtained is taken as u_{n+1}.

7.2.5. Finding solutions with boundary conditions identification. The main procedure of the proposed algorithm is solution of the system (7.12) for fixed u. It encompasses the Newton iteration as a method for minimization of the discrepancy. When the approximation $\mathbf{x}_u(t)$, $\mathbf{y}_u(t)$ is found, the solutions of the system are (7.12). From the point $\mathbf{x}_u(t)$, $\mathbf{y}_u(t)$ carry out one step of the Newton method and receive a new approximation $\mathbf{x}'_u(t)$, $\mathbf{y}'_u(t)$ as a solution for the linear system

$$\mathbf{x}'_u(t) - \mathbf{x}'_u(t-1) = \mathbf{F}^*\big(\mathbf{z}_u(t)\big) + \frac{\partial \mathbf{F}^*}{\partial \mathbf{x}}\big(\mathbf{z}_u(t)\big) \cdot \big(\mathbf{x}'_u(t) - \mathbf{x}_u(t)\big) + \frac{\partial \mathbf{F}^*}{\partial y}\big(\mathbf{z}_u(t)\big) \cdot \big(\mathbf{y}'_u(t) - \mathbf{y}_u(t)\big),$$
$$0 = \mathbf{G}^*\big(\mathbf{z}_u(t)\big) + \frac{\partial \mathbf{G}^*}{\partial \mathbf{x}}\big(\mathbf{z}_u(t)\big) \cdot \big(\mathbf{x}'_u(t) - \mathbf{x}_u(t)\big) + \frac{\partial \mathbf{G}^*}{\partial \mathbf{y}}\big(\mathbf{z}_u(t)\big) \cdot \big(\mathbf{y}'_u(t) - \mathbf{y}_u(t)\big),$$
$$\mathbf{z}_u(t)\langle \mathbf{x}_u(t), \mathbf{y}_u(t), \mathbf{s}_u(t)\rangle, \quad t = t0+1, t0+2, \ldots, T.$$

This system can be written beforehand, and $y'_u(t)$ can be analytically excluded from the second equation. Then in order to determine $x'_u(t)$, $y'_u(t)$ we get the system

$$\mathbf{x}'_u(t-1) = \mathbf{A}\big(\mathbf{z}_u(t)\big) \cdot \mathbf{x}'_u(t) + \mathbf{f}\big(\mathbf{z}_u(t)\big), \quad \mathbf{y}'_u(t) = \mathbf{B}\big(\mathbf{z}_u(t)\big) \cdot \mathbf{x}'_u(t) + \mathbf{g}\big(\mathbf{z}_u(t)\big), \quad t = t0, \ldots, T. \tag{7.13}$$

The system (7.13) lacks $\dim(\mathbf{x}'_u)$ boundary conditions, which means that one step of the Newton method provides not only one point but a hyperplane of dimension $\dim(\mathbf{x}'_u)$ in the space of time series of variables of the model.

A point on this hyperplane is chosen as a projection onto this hyperplane of some point $\mathbf{m}_u$ in the space of time series of model variables. The point $\mathbf{m}_u$ is set in the form of $\mathbf{m}_u = (1-u)\bar{\mathbf{x}}^* + u\mathbf{m}$, where $\mathbf{m}$ is a set of variable values that best fits statistical data, and $\bar{\mathbf{x}}^*$ is initial balances growth.

The next approximation is considered unsuccessful if the discrepancy of $\mathbf{x}'_u(t)$, $\mathbf{y}'_u(t)$ increases greatly compared to that of the previous approximation $\mathbf{x}_u(t)$, $\mathbf{y}_u(t)$. In case of failure the homotopy parameter u is decreased.

The process of finding a solution is considered successful if the discrepancy is small enough and decreases fast enough within the steps of the Newton method under fixed boundary conditions. The described procedure has a significant feature. The system of difference equations in (7.13) is highly unstable. The expression $\mathbf{x}'_u(t0)$ in terms of $\mathbf{x}'_u(T)$ has coefficients of order about $10^{50} \div 10^{70}$ which appear as a result of multiplication and addition of values of degree one. To determine high degree values through low degree values we have to conduct calculations with high precision. Commonly a precision of 100 decimal places suffices.

However, the obtained expressions cannot be used for minimization of the discrepancies because the gradients are too big. Hence, prior to minimization the solution is regularized: out of $2\dim(\mathbf{x}'_u)$ of initial and terminal values $\mathbf{x}'_u(t)$ we choose those $\dim \mathbf{x}'_u(t)$ which express other values with coefficients of degree 1. This can always be done if the linear manifold determined by the system (7.13) is close to zero. The expression through such parameters implies that we consider only finite solutions of the system (7.13).

7.3. The Results of Calculations of the Model of the Russian Economy. The calculations are performed using the quarterly data from the first quarter of 2004 until the second quarter of 2010. In total there are 30 adjustment parameters, whose values cannot be found in official statistical data (e.g., the parameter for loans write-offs, the Household's stocks depreciation μ, and so on). These parameters are used to optimize the solution, i.e., we try to find a set of parameters that fits the solution to the statistical data as close as possible. In order to find optimal sets of parameters we applied optimization algorithms on mainframe MVS100-K (140.16 TFlops) of JSCC (Joint Supercomputer Center of RAS).

Figures 7.3.1–7.3.14 show the calculation results for the main observable indicators (solid line is official statistical data, dotted line represents calculated values). The main benefit of the model is that it is capable of reproducing the phenomenon of crisis in the economy.

One can see that the model reproduces dynamics of the main real indicators: GDP (Fig. 7.3.1), gross Investments (Fig. 7.3.3), Consumption (Fig. 7.3.5), Exports (Fig. 7.3.7), and Imports (Fig. 7.3.8). These pictures reveal that the model reproduces both phases of oscillation and crisis decline.

The model also reproduces dynamics of price indexes (the deflators): GDP (Fig. 7.3.2), gross Investments (Fig. 7.3.4), and Consumption (Fig. 7.3.6). The model vividly reveals the difference between the Russian crisis and the crisis in other countries: the decline in total production is accompanied by deflations worldwide, but in Russia the decline in this crisis period is accompanied by inflation. However the model reproduces (though with excess "splashes") the only deflation process observed in Russia—the gross Investments price index decline at the end of year 2008.

The model also reproduces the dynamics of the main financial indicators: loans (Fig. 7.3.9), savings (Fig. 7.3.10), liquidity (Fig. 7.3.11, 7.3.12), and interest rates (Fig. 7.3.13).

Figure 7.3.14 shows that the model reproduces a unique feature of the Russian banking system: the "excess money phenomenon." Unlike commercial banks of other countries Russian banks do not borrow money from the Central Bank. Russian banks lend money to the Central Bank at a low interest rate. It was not until the end of year 2008 that the Russian banks borrowed sufficient funds from the Central Bank and became a net borrower. However in the model these borrowings are described as an additional demand from Banks for borrowed funds, which was caused by the crisis.

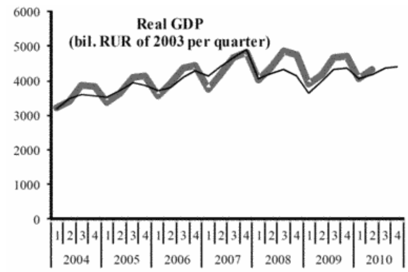

Fig. 7.3.1

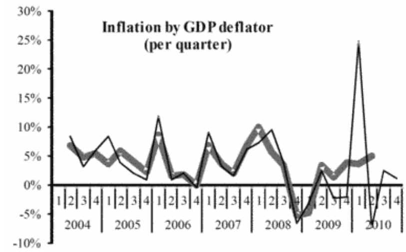

Fig. 7.3.2

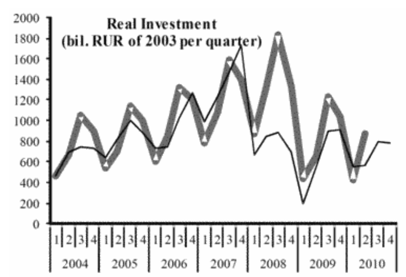

Fig. 7.3.3

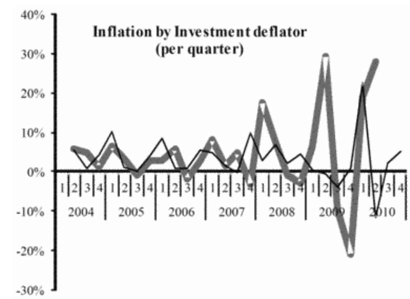

Fig. 7.3.4

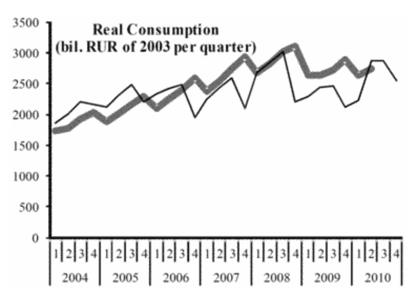

Fig. 7.3.5

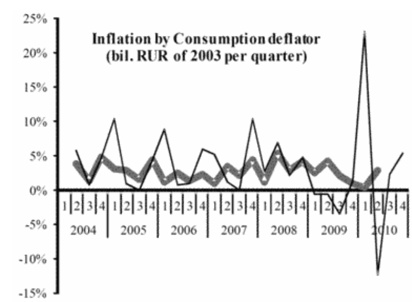

Fig. 7.3.6

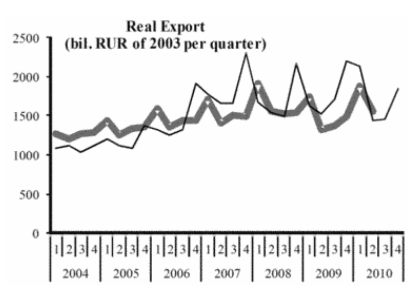

Fig. 7.3.7

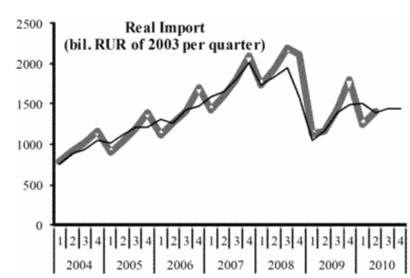

Fig. 7.3.8

7.4. Strong Turnpike Property. The most important and interesting result of our studies for the last two years was the revelation that after an identification of parameters the model equations from the mathematical point of view *are not in the general position.* For instance, in the model of Bank (Sec. 6.3) under certain parameters, which make the model to reproduce the statistical data, the turnpike property is manifested so strong that the impact of the future shocks is faded one step backwards [7].

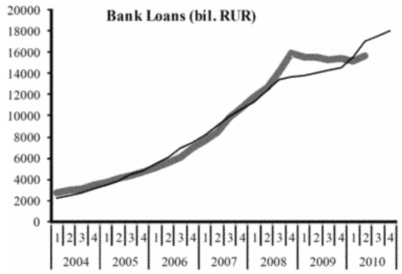

Fig. 7.3.9

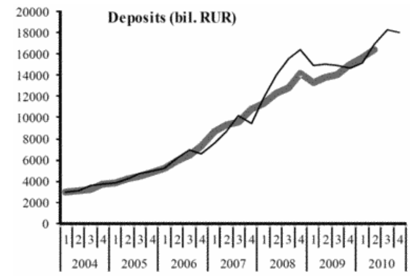

Fig. 7.3.10

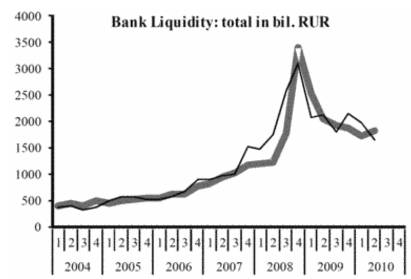

Fig. 7.3.11

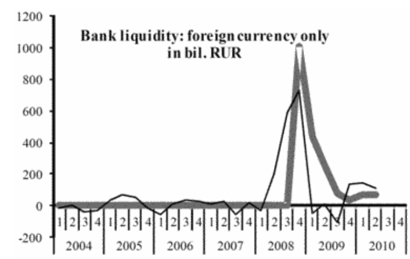

Fig. 7.3.12

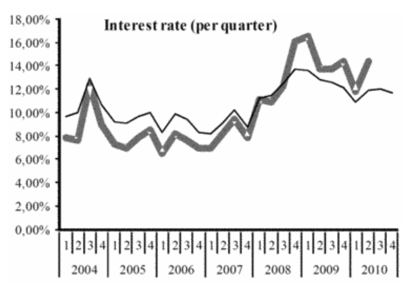

Fig. 7.3.13

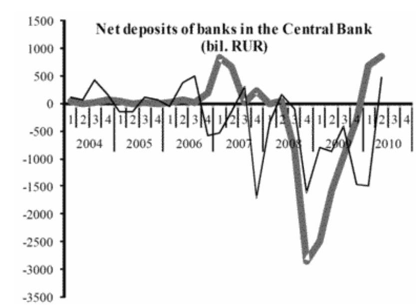

Fig. 7.3.14

From the mathematical standpoint it means that the agent's behavior is described by a dynamic system. Substantially it means that although we allow for an agent to know the future, the institutional constraints combined with good parameters identification "restrict" the agent's possibilities so tightly that *in order to derive optimal solution it suffices to know only the current economic situation.* Nevertheless, we have to state the problem for an agent and derive its optimality conditions (regardless of the turnpike property) because otherwise we cannot receive expressions for important variables, such as interior return.

For some versions of the complete model of the Russian economy the turnpike property was revealed. Unfortunately, the version of the model described in this paper does not reveal this property strongly enough. For this reason, we do not perform analytical and forecast calculations. Based on the previous experience we expect to find a set of parameters to reveal the strong turnpike property. Then the model would be completely workable.

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