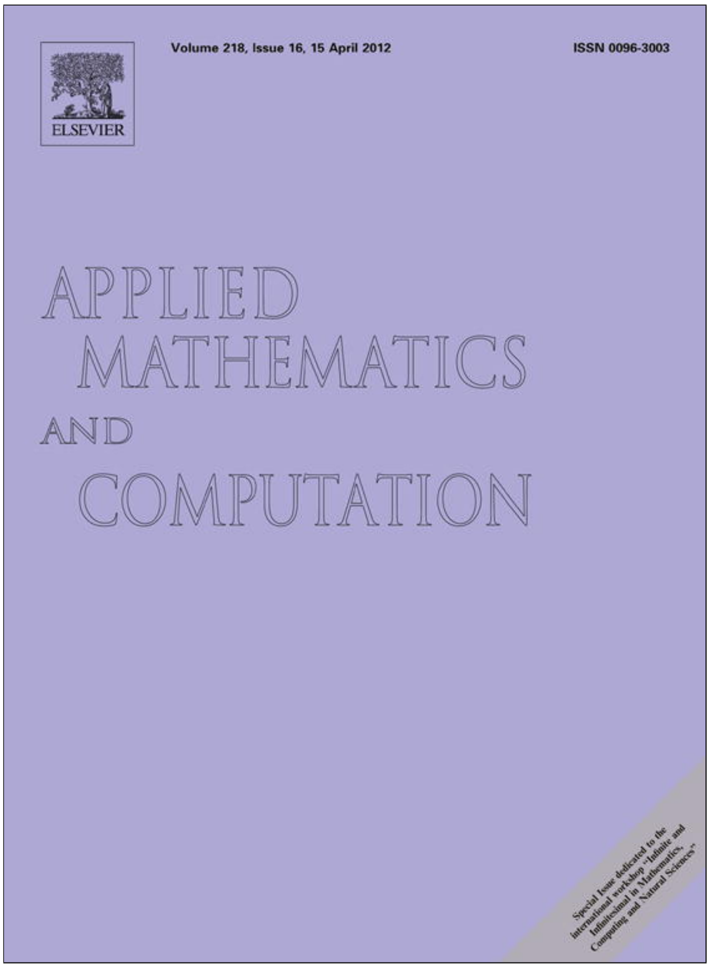
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Defaults and infinite prices in a stochastic pure exchange model

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ABSTRACT

A stochastic pure exchange model with perfect foresight is considered. Real indeterminacy arises in this setting in the form of continuum of equilibria corresponding to different commodity allocations. The cause of real indeterminacy is arbitrariness of prices at the final date. If some arbitrary price tends to infinity then the equilibrium in the limit has an intuitive economic interpretation. The limiting equilibrium with formally infinite prices we interpret as an equilibrium with default: if a special state of the environment occurs, consumers may forget about their debts and savings and start new life from scratch.

Existence of equilibria with defaults is proven. A numerical experiment shows that in some cases equilibria with finite prices are Pareto dominated by equilibria with defaults.

1. Introduction

General equilibrium models provide an approach to describe defaults. Default is considered as an event in which an agent refuses to pay her debt. Even in the case when agents are able to foresee future events there is a way to introduce the possibility of default. This can be done by, for example, imposing a certain penalty on an agent in the case of refusal to pay the debt. This punishment may take the form of loss of collateral [1] or disutility from default [2]. In the purely theoretical model to be considered here, there is no debt penalty and all agents foresee the possibility of default.

Formally, the model considered in this paper is a variation of the Arrow–Debreu model [3] with incomplete markets. This is a relatively simple stochastic pure exchange framework with perfectly foreseeing agents. There is a set of commodities that are perishable and could not be saved for future. No production, firms and financial instruments is assumed. There are only consumers with random endowments, who face intertemporal budget constraints of non-negativity of the "money".

There is a continuum of equilibria in the model considered. Equilibrium allocations depend on equilibrium prices continuously. This fact is usual for the Arrow–Debreu models and is known as *indeterminacy* [4]. In the case of incomplete markets the indeterminacy is *real* [5], that is, different equilibrium prices lead to distinct equilibrium allocations. The reason for real indeterminacy is arbitrariness of prices at the final date. Equilibrium exists for any set of the prices at the final date, see Proposition 1. This is so called Walras' law. Among others, equilibria with defaults arise when some price at the final date tends to infinity.

The idea of infinite prices is not unusual for mathematical economics. For example, this idea (in the terms of delta functions) was used to describe peak-load pricing on the electric energy market [6], or to describe the agents' attitude to catastrophes and rare events [7]. We generalize the notion of "price" to an exchange ratio analogous to [8] to describe the equilibria with infinite prices.

The paper proceeds as follows. In Section 2 we describe a pure exchange model with finite prices. Section 3 provides an example and illustrates real indeterminacy in a simple case. A generalization of pure exchange framework that aims to

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encompass infinite prices is presented in Section 4. In Section 5 we select and describe equilibria with defaults and discuss some properties for them.

2. Stochastic pure exchange model with finite prices

Stochastic pure exchange models are often constrained to the case of two periods of time. Here we formulate a pure exchange model for any finite number of periods of time. This would demand some additional notations and definitions. In order to provide some intuition on the mechanism of the model, we analyze a standard two-period numeric example in the next section.

We consider an economy that consists of a finite number of *consumers* indexed by $n \in \mathcal{N}$. Consumers live during a finite number of periods of time $t \in \{0,\ldots,T\}$. There is a single commodity in the model. Each consumer receives a (random) *endowment* of this commodity at each date t. The commodity is perishable and brings consumption utility. It is not considered as money, but rather as some kind of real goods like coconuts that fall from coconut palms. If attitudes toward risk and endowments vary across agents, then exchange of the commodity may be mutually beneficial. Consumers could trade the coconuts-today for the coconuts-tomorrow.

Further we use the notations of Radner [9]. We follow the perfect foresight principle and assume that all consumers have the same (symmetric) information about the environment.

The following notation is used throughout the paper. Let s_t denote a *state of environment* at the date t, and let a sequence $s = (s_0,\ldots,s_T)$ denote a *complete history* of the environment. The set of alternative states s_t at date t is denoted by S_t. We assume that each set S_t is finite. Let the *elementary events* at date t be the sequences $e_t = (s_0,\ldots,s_t)$. By $\mathcal{E}$ we denote the set of all elementary events e_t, and by S – the set of all complete histories s.

There is a natural partial ordering on $\mathcal{E}$. We denote this order by $e_{t'} \leqslant e_{t''}$: an elementary event $e_{t''}$ is better than $e_{t'}$ if $e_{t'}$ is an initial subsequence of $e_{t''}$.

Let $w^n(e_t)$ denote consumer n's endowment of the commodity at an elementary event e_t. By $c^n(e_t)$ consumer n's consumption of the commodity at an elementary event e_t is denoted. In accordance with the perfect foresight approach we assume that all consumers agree on what prices correspond to each future event. In the same manner $p(e_t)$ denotes the *price* of the commodity at an elementary event e_t. Endowments and consumption are non-negative, and prices are positive.

For a given complete history s in the set S, let $p(s), w^n(s), c^n(s)$ denote the realized history of prices, consumer n's endowments and consumption, respectively. These are vectors in a space of dimension $T+1$: $p(s) \in \mathbb{R}^{|T+1|}_{++}, w^n(s) \in \mathbb{R}^{|T+1|}_{+}, c^n(s) \in \mathbb{R}^{|T+1|}_{+}$. We also denote the system of prices, consumer n's endowments and consumption by $p = (p(e_t)), w^n = (w^n(e_t)), c^n = (c^n(e_t))$, where the variable e_t takes values from the set $\mathcal{E}$. These are vectors in a space of dimension $|\mathcal{E}|$: $p \in \mathbb{R}^{|\mathcal{E}|}_{++}, w^n \in \mathbb{R}^{|\mathcal{E}|}_{+}, c^n \in \mathbb{R}^{|\mathcal{E}|}_{+}$.

Consumer chooses some personal *consumption plan* c^n at date $t = 0$. We suppose for simplicity, that there is no uncertainty about the state of environment at the initial date $t = 0$, i.e. S_0 consists of only one state. The plan $c^n \in \mathbb{R}^{|\mathcal{E}|}_{+}$ is *feasible* if it satisfies intertemporal *budget constraints*

$$p(s)c^n(s) \leqslant p(s)w^n(s), \quad s \in S. \tag{2.1}$$

Note that the number of these budget constraints is equal to the number of complete histories s.

For simplicity of notation, we combine the consumer n's set of feasible consumption plans c^n into *consumption set* denoted $B^n(p)$.

Each consumer has a preference pre-ordering on the consumption set $B^n(p)$. Preference pre-ordering is assumed to be induced by an *expected utility* $\mathsf{E}^n\left\{\sum_{t=0}^{T} U^n(c^n(e_t))\right\}$ of a consumption plan c^n, where $U^n : \mathbb{R}_+ \to \mathbb{R}_+$ is some *utility function*.

The *expectation operator* E^n is indexed by n in order to take into account the fact that that different consumers may have different probability measures over the set S of complete histories s. These probability measures reflect personal beliefs at the initial date $t = 0$ concerning the relative likelihoods of complete histories. In general, these measures may vary across agents. The only restriction we impose is that the consumer n's subjective probability of any complete history s should be positive.

Definition 1. *Proper equilibrium* or *equilibrium with finite prices* is a combination $(p, c^{\mathcal{N}})$ of prices $p \in \mathbb{R}^{|\mathcal{E}|}_{++}$ and consumption $c^{\mathcal{N}} = \langle c^n | n \in \mathcal{N}\rangle$ such that

(1) for each consumer $n \in \mathcal{N}$ the consumption plan c^n maximizes the expected utility over the consumption set $B^n(p)$,
(2) material balances hold true

$$\sum_{n \in \mathcal{N}} w^n(e_t) \geqslant \sum_{n \in \mathcal{N}} c^n(e_t), \quad e_t \in \mathcal{E}. \tag{2.2}$$

The term "proper" in Definition 1 reflects the fact that all prices under consideration are finite.

If the consumption utility function is non-satiated, then one should expect the budget constraints (2.1) hold as equalities in equilibrium. Prices should be positive for the same reason. Then the material balances (2.2) in equilibrium will also hold as equalities. If one takes sum of (2.1) over consumers $n \in \mathcal{N}$ or multiplies (2.2) by $p(e_t)$ and for a given s takes sum over states $e_t | e_t \leqslant s$ then one may obtain the same result: $p(s)\sum_{n\in\mathcal{N}} c^n(s) = p(s)\sum_{n\in\mathcal{N}} w^n(s)$ (Walras' law). As a result, the budget constraints in (2.2) for any elementary event e_T at the final date T are redundant. Appropriate prices $p(e_T)$ at the final date T could not be determined from the budget constraints (2.2). Thus, one should expect that proper equilibrium exists for any prices $p(e_T), e_T \in \mathcal{E}$.

We denote the set of prices $p(e_T), e_T \in \mathcal{E}$ at the final date T by a vector p_T, and we will call this set *terminal prices*. It is easy to see that the vector p_T has dimension equal to $|S|$. The set of prices excluding p_T we denote by $p \backslash p_T$:

Proposition 1. *Let*

(a) *the endowments of each consumer $n \in \mathcal{N}$ be strictly positive: $w^n(e_t) > 0, e_t \in \mathcal{E}$;*
(b) *the utility function U^n for each consumer $n \in \mathcal{N}$ be smooth, concave and satisfy the condition:*

$$\partial U^n(0) = \infty; \quad \partial U^n(x) > 0, \quad \partial^2 U^n(x) < 0, \quad x > 0.$$

Let $q \in \mathbb{R}^{|S|}_{++}$ be an arbitrary positive vector of dimension $|S|$.

Then there exists a proper equilibrium $(p, c^{\mathcal{N}})$ such that $p_T = q$.

The proof of Proposition 1 is standard, so we have only sketched it in Appendix A.

3. An example of real indeterminacy

As we mentioned above different terminal prices p_T (see Proposition 1) may lead to different equilibria $(p, c^{\mathcal{N}})$. This fact is known as real indeterminacy. One may illustrate real indeterminacy on one simple example. Consider the case of two consumers $n \in \{1,2\}$, that act for two periods of time $t \in \{0,1\}$, with the initial state, $s_0 = 1$, and two possible states $s_1 \in \{1,2\}$ in the second ($t = 1 = T$) period of time. Beliefs of the consumers are assumed to coincide. The probabilities of the states $\{1,2\}$ at the date $t = 1$ are π_1 and $\pi_2 = 1 - \pi_1$, respectively.

The elementary events in this case are: $e_0 = 1$, $e_1 = (s_0, s_1) \in \{(1,1),(1,2)\}$. Then, the problem of each consumer $n \in \{1,2\}$ is the following: consumer chooses consumption $c^n(1)$, $c^n(1,1)$, $c^n(1,2)$ in order to maximize the expected utility

$$U^n(c^n(1)) + \pi_1 U^n(c^n(1,1)) + \pi_2 U^n(c^n(1,2))$$

subject to budget constraints

$$p(1)(c^n(1) - w^n(1)) + p(1,1)(c^n(1,1) - w^n(1,1)) \leqslant 0, \quad p(1)(c^n(1) - w^n(1)) + p(1,2)(c^n(1,2) - w^n(1,2)) \leqslant 0.$$

The material balances are

$$\sum_{n\in\{1,2\}} w^n(e_t) \geqslant \sum_{n\in\{1,2\}} c^n(e_t), \quad e_t \in \{1,(1,1),(1,2)\}.$$

As stated in Proposition 1, two terminal prices $p(1,1)$, $p(1,2)$ are arbitrary, i.e. for any positive prices $p(1,1)$, $p(1,2)$ there exists an equilibrium. As a result, the dimension of the real indeterminacy in this case is 2. However, one dimension is only nominal: all prices $p(1)$, $p(1,1)$, $p(1,2)$ could be multiplied by the same positive multiplier without any change in the equilibrium allocation of the commodity. Therefore, the dimension of real indeterminacy is in fact one. Geometrically, one should expect that the equilibria form a line in the space of the agents' expected utilities $\left[\mathsf{E}\left\{\sum_{t=0}^{1} U^1(c^1(e_t))\right\}, \mathsf{E}\left\{\sum_{t=0}^{1} U^2(c^2(e_t))\right\}\right]$. The proper equilibria for all possible terminal prices $p(1,1)$, $p(1,2)$ are marked on Figs. 1 and 2 by small crosses. The equilibria with infinite prices (to be more precise, since we have not provided strict definition yet, when one of the terminal prices, $p(1,1)$ or $p(1,2)$, tends to infinity) are marked by big crosses. Circles in the right upper corner of the figures represent the solutions of the corresponding welfare problem. The corresponding welfare problem (for the case when beliefs of consumers coincide) is a maximization of the functional $\sum_{n\in\mathcal{N}} a_n \mathsf{E}^n\left\{\sum_{t=0}^{T} U^n(c^n(e_t))\right\}$, subject to the constraint (2.2).

The utility functions and the sums of endowments $\sum_{n\in\mathcal{N}} w^n(e_t)$, $e_t \in \mathcal{E}$ are the same for both Figs. 1 and 2: $U^1(x) = B^2 - (B - x)^2$, $U^2(x) = 0.5\sqrt{x}$, $\sum_{n\in\{1,2\}} w^n(1) = 1$, $\sum_{n\in\{1,2\}} w^n(1,1) = 0.48$, $\sum_{n\in\{1,2\}} w^n(1,2) = 0.5$. Figs. 1 and 2 differ only in agents' endowments: $w^1(1) = 0.7$, $w^1(1,1) = 0.336$, $w^1(1,2) = 0.35$ for Fig. 1 and $w^1(1) = 0.6$, $w^1(1,1) = 0.144$, $w^1(1,2) = 0.35$ for Fig. 2. Remaining parameters are: $B = 1.9$, $\pi_1 = 0.55$, $\pi_2 = 0.45$.

The equilibria with infinite prices are dominated by the equilibria with finite prices on Fig. 1. However, the equilibria with infinite prices dominate the equilibria with the finite prices on Fig. 2. A conclusion suggests itself that in some cases the equilibria with infinite prices are turn out to be Pareto optimal. In other words, it may be "more profitable" for the consumers to believe consistently that the prices will be infinite.

A series of computations of equilibria in the cases of the proposed model with more consumers, more states of nature and with initial debts and savings Φ_0^n, $n \in \mathcal{N}$, $\sum_{n\in\mathcal{N}} \Phi_0^n = 0$ is also conducted. Naturally, the figures are more complicated. Nevertheless the result stays the unchanged: in some instances the equilibria with infinite prices dominate over the equilibria with the finite prices.

Further we present a generalization of the stochastic pure exchange model in order to formulate the equilibria with infinite prices correctly. In Section 5, we give an interesting interpretation of the equilibria with infinite prices.

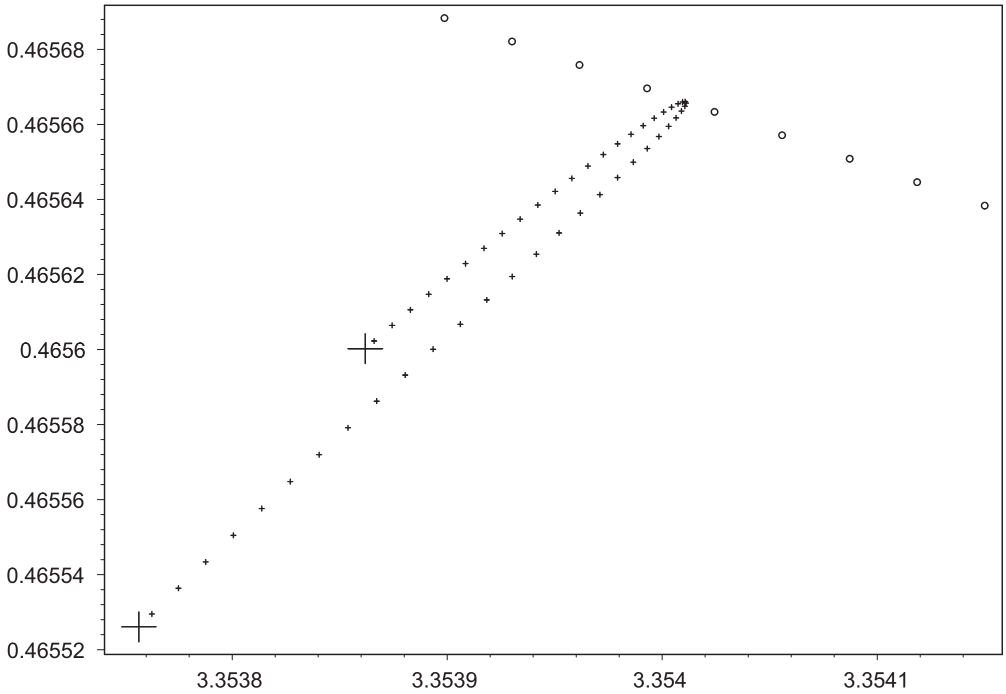

Fig. 1. The equilibria with finite prices (small crosses) in the space of agents' expected utilities. Example 1: The equilibria with infinite prices (big crosses) are dominated by the equilibria with the finite prices.

4. Stochastic pure exchange model with actually infinite prices

We can rewrite the budget constraints (2.1) in the form of sum

$$\sum_{e_t | e_t \leqslant e_T} p(e_t) c^n(e_t) \leqslant \sum_{e_t | e_t \leqslant e_T} p(e_t) w^n(e_t), \quad e_T \in \mathcal{E}.$$

The sum is taken over all elementary events e_t which are subsequences of a complete history e_T (note that there is a correspondence between the elementary events e_T at the final date T and complete histories $s \in \mathcal{S}$). The number of the summands

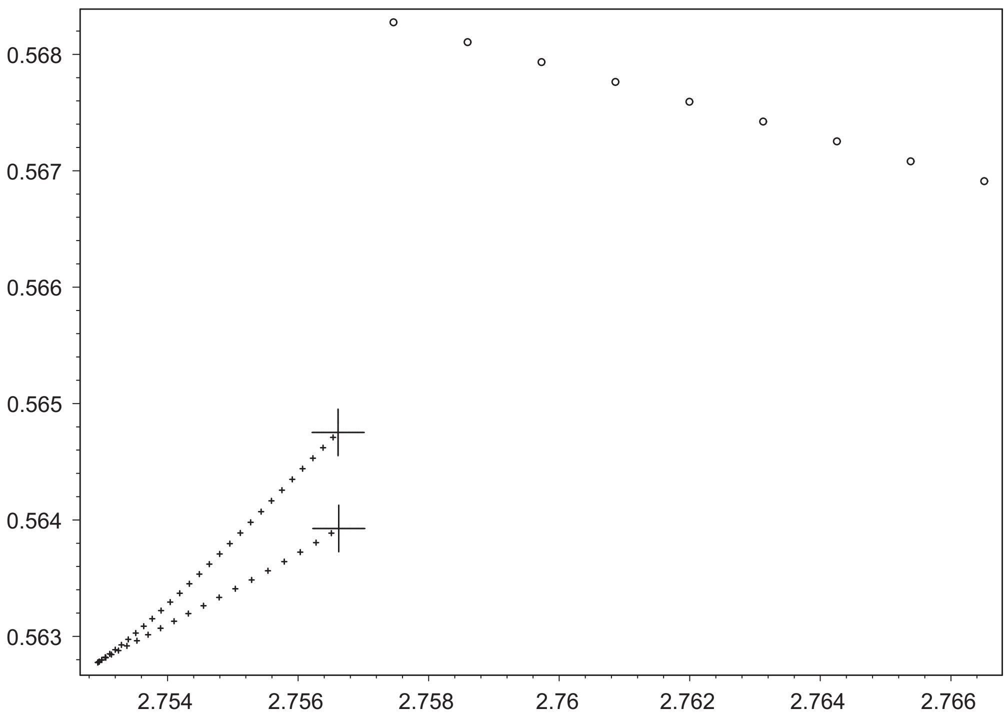

Fig. 2. The equilibria with finite prices (small crosses) in the space of agents' expected utilities. Example 2: equilibria with infinite prices (big crosses) dominate the equilibria with the finite prices.

in are $T+1$. In order to deal with infinite prices we divide each budget constraint by the maximum price $p(e_{t'}) \in Arg\max\{p(e_t)|e_t \leqslant e_T\}$ over the complete history $e_T : \sum_{e_t|e_t\leqslant e_T}\frac{p(e_t)}{p(e_{t'})}c^n(e_t) \leqslant \sum_{e_t|e_t\leqslant e_T}\frac{p(e_t)}{p(e_{t'})}w^n(e_t)$.

In case one of the prices tends to infinity, nevertheless, the ratios $p(e_t)/p(e_{t'})$ remain finite and the budget constraints play their roles. This idea suggests using of the price ratios $p(e_t)/p(e_{t'})$ as some new analog prices.

Following Danilov and Sotskov [8] we define *comparative prices* $P(e'_{t'}, e''_{t''}), e'_{t'} \in \mathcal{E}, e''_{t''} \in \mathcal{E}$ over the pairs of elementary events. Each comparative price determines what amount of the commodity at an elementary event $e''_{t''}$ could be traded for one unit of the commodity at some other elementary event $e'_{t'}$. Thus comparative prices may be considered as **exchange ratios**.

A comparative price may be a nonnegative number or infinity. In addition, comparative prices should satisfy a natural transitivity condition:

$$P(e'_{t'}, e''_{t''})P(e''_{t''}, e'''_{t'''})P(e'''_{t'''}, e'_{t'}) = 1. \tag{4.1}$$

Remark. If one of the comparative prices in (4.1) is infinite, then (4.1) is valid by definition if and only if another comparative price is equal to 0.

Thus Eq. (4.1) does not define a multiplication operation of the prices in the case when one price is zero and another is infinity. Eq. (4.1) is only a necessary condition to define a *hierarchic price*? as a set of comparative prices:

$$\mathcal{P} = \langle P(e'_{t'}, e''_{t''})|e'_{t'} \in \mathcal{E}, e''_{t''} \in \mathcal{E},\ (4.1)\rangle.$$

Comparative prices generate some ordering relation on the set of elementary events $\mathcal{E}$. In our setting we define a relation in which $e'_{t'} \succ^P e''_{t''}$ (*not worse*) if and only if $P(e'_{t'}, e''_{t''}) > 0$.

Relation $\succ^P$ is complete. A corresponding equivalence relation, given by $e'_{t'} \approx e''_{t''} \iff \infty > P(e'_{t'}, e''_{t''}) > 0$, partitions the set of the events $\mathcal{E}$ into a finite number L of disjoint *equivalence classes* $K_1, \ldots, K_L$. Classes K_i, $i = 1, \ldots, L$ are ordered: K_1 is the class of the worst events (with the lowest prices), and K_L is the class of the best events (the highest prices).

Let *rank* $j(e_t)$ be the index of the event e_t class, i.e. $e_t \in K_{j(e_t)}$. Then one may conclude that $e'_{t'} \succ^P e''_{t''} \iff j(e'_{t'}) \geqslant j(e''_{t''})$.

We consider the same single commodity economy as in Section 2, with a finite set $\mathcal{N}$ of consumers indexed by $n \in \mathcal{N}$, the set of the elementary events $\mathcal{E}$, etc.

From now, we call a *consumption set* of a consumer $n \in \mathcal{N}$ the following set:

$$B^n(\mathcal{P}) = \left\{c^n \in \mathbb{R}^{|\mathcal{E}|}_+ | \sum_{e_t|e_t\leqslant e_T} P(e_t, e_{t'})c^n(e_t) \leqslant \sum_{e_t|e_t\leqslant e_T} P(e_t, e_{t'})w^n(e_t), j(e_{t'}) = \max_{e_t|e_t\leqslant e_T}\{j(e_t)\},\ e_T \in \mathcal{E}\right\}. \tag{4.2}$$

Note that, as before, the definition of the consumption set (4.2) consists of $|S|$ budget constraints. For each complete history $s \in S$ and for an appropriate budget constraint we choose an event $e_{t'}$ of the highest rank among the events $e_t \leqslant e_T$. We use the selected event $e_{t'}$ as the second argument of all comparative prices $P(e_t, e_{t'})$ in the budget constraint. As a result, **all comparative prices in (4.2) are finite**, with at least one price equal to 1, and possibility of some prices to be zero.

An improper equilibrium or *equilibrium with the infinite prices* is a bundle $(\mathcal{P}, c^{\mathcal{N}})$ of the hierarchic price $\mathcal{P}$ and consumption $c^{\mathcal{N}} = \langle c^n | n \in \mathcal{N}\rangle$ such that.

(1) $c^n \in Arg\max_x \left\{E^n\left\{\sum_{t=0}^{T} U^n(c^n(e_t))\right\}|x \in B^n(\mathcal{P})\right\}, \quad n \in \mathcal{N},$

(2) the material balances (2.2) hold true.

The existence result for the improper equilibrium is analogous to Proposition 1 from Section 2.

Proposition 2. *Let $\mathcal{Q} = \{Q(e'_T, e''_T)|e'_T \in \mathcal{E}, e''_T \in \mathcal{E}\}$ be some arbitrary set of the comparative prices defined on pairs of elementary events at the final date T and satisfying (4.1). Then under assumptions* (a) *and* (b) *of Proposition 1 there exists an improper equilibrium $(\mathcal{P}, c^{\mathcal{N}})$ such that $\mathcal{P}_T = \mathcal{Q}$, where $\mathcal{P}_T = \{P(e'_T, e''_T) \in \mathcal{P}|e'_T \in \mathcal{E}, e''_T \in \mathcal{E}\}$.*

The proof of Proposition 2 is provided in Appendix B.

5. Equilibria with defaults and their properties

As one may see in this setting the set of possible improper equilibria includes the proper equilibria. If an improper equilibrium is not a proper equilibrium, then we call it an *equilibrium with defaults*. In other words, the equilibrium with defaults is an equilibrium with infinite prices in the case when prices are indeed infinite. The explanation of the term "default" will be given below.

The equilibria are illustrated on Figs. 1, 2 where the proper equilibria are marked by small crosses, and the equilibria with defaults are marked with big crosses. This example considers a two-period model. The equilibria with defaults on Fig. 1 correspond to comparative prices $P((1),((1),(1,1))) = 0$, $P((1),((1),(1,2))) = 0.593$ and $P((1),((1),(1,1))) = 0.713$, $P((1),((1),(1,2))) = 0$. For Fig. 2 the comparative prices are $P((1),((1),(1,1))) = 0$, $P((1),((1),(1,2))) = 0.579$ and $P((1),((1),(1,1))) = 0.695$, $P((1),((1),(1,2))) = 0$.

This begs a question whether the equilibria with defaults have any economic sense? Or they are merely mathematical objects? Is the model of economy arranged rationally? Next statements provide some economic interpretation.

Property 1. *If* $(\mathcal{P}, c^{\mathcal{N}})$ *is an improper equilibrium, then for any complete history s the ranks of the elementary events on that complete history do not decrease with time:* $e'_{t'} \leqslant e''_{t''} \Rightarrow j(e'_{t'}) \leqslant j(e''_{t''})$.

From the viewpoint of a consumer, who acts sequentially during the periods from $t = 0$ to $t = T$ and observes "finite" prices $p(e_t)$ (in the terms introduced in Section 2), Property 1 means that the prices $p(e_t)$ could rise to infinity from time to time, but prices can not fall to zero.

We could rewrite each budget constraint (see (4.2)) in the form $\sum_{e_t|e_t \leqslant e_T, t=t''',ldots,T} P(e_t, e_{t'})(w^n(e_t) - c^n(e_t)) \geqslant 0$: for any complete history s all comparative prices are zeroes before some date t''' ($t''' = 0$ is possible), and all comparative prices are positive and finite after the date t'''.

Let some comparative price from some budget constraint equal to zero. Without loss of generality assume $P(e_0, e_{t'''}) = 0$. Is it possible that the consumption $c^n(e_0)$ is unrestricted? The answer turns out to be negative. It is impossible because there always exists another budget constraint and appropriate complete history s' with $P(e_0, e_{t'}) > 0,\ e_{t'} \leqslant s'$. This situation is excluded due to the following Property 2.

Property 2. *If* $(\mathcal{P}, c^{\mathcal{N}})$ *is an improper equilibrium, then the rank of every non-terminal elementary event* e_t, $t < T$ *equals to the minimal rank of the subsequent events:* $e_t \in \mathcal{E},\ t < T \Rightarrow j(e_t) = \min_{e_{t+1}|e_t \leqslant e_{t+1}} \{j(e_{t+1})\}$.

It is easy to prove both properties from the contrary. Contradiction arises from consumer's desire to buy unlimited amount of commodity in the events with low prices.

Properties 1 and 2 allow determining the rank of any non-terminal state e_t, $t < T$ by the terminal comparative prices $\mathcal{P}_T$ **uniquely**.

From the viewpoint of a consumer, who lives at the date $t = 0$, the exchange ratios could fall to zero or stay comparable at the date $t = 1$. Assume the exchange ratio falls to zero in the state $s_1 = 1$, i.e. $P((s_0),((s_0),(s_0,1))) = 0$, and stays comparable in the state $s_1 = 2$, $\infty > P((s_0),((s_0),(s_0,2))) > 0$. If the state $s_1 = 2$ takes place and $w^n(e_0) - c^n(e_0) > 0$ then one may say that the consumer n made savings after the date $t = 0$ and before the date $t = 1$. If $s_1 = 2$ occurs and $w^n(e_0) - c^n(e_0) < 0$, then the consumer n had debt. If $s_1 = 1$ occurs then the consumer had neither debts nor savings, because natural debts or savings $w^n(e_0) - c^n(e_0)$ had been devalued by the zero comparative price (or by the infinite prices $p((s_0),(s_0,1)) = \infty$ in the terms of the "finite" prices of the Section 2). After devaluation, the consumer begins financial life from scratch. This particular situation is named by the term "default" in our framework. Since all consumers foresee the prices, refusal to pay the debts $w^n(e_0) - c^n(e_0)$ will be mutual.

As one may see from Fig. 2, the equilibria with defaults (big crosses) could dominate all equilibria with finite prices. Therefore a default may be a favorable event.

At the present time, strict conditions for dominance of the equilibria with defaults still remain a subject of research. Nevertheless, some qualitative condition may be outlined. For this, let the endowments be independent of the states s_t at each date t and for each consumer $n \in \mathcal{N} : w^n(e_t) = w^n_t$. In addition, let consumers still believe that the prices $p(e_t)$ depend on the states s_t (thus, the prices depend on elementary events e_t). This type of setting is known as a model *with sunspot equilibria* [10]. The terminal prices $p(e_T)$ still may be arbitrary (due to Proposition 1), so real indeterminacy still exists. In these conditions one may consider a certain equilibrium with equal terminal prices: $p(e_T) = p_T,\ e_T \in \mathcal{E}$. As a result, uncertainty does not play any role, and all prices and consumption depend only on time: $c^n(e_t) = c^n_t, p(e_t) = p_t$. Thus, the equilibrium is deterministic. Since it is well known that the equilibrium in a deterministic pure exchange model is Pareto efficient, the equilibrium with the prices $p(e_T) = p_T,\ e_T \in \mathcal{E}$ could not be dominated by any other sunspot equilibrium. Consequently, Pareto dominance of equilibria with defaults is possible only if the endowments depend on the states of the environment s_t explicitly.

6. Conclusion

In this paper a stochastic model with incomplete markets is considered. Author illustrates appearance of real indeterminacy in this setting. The model considered here is based on the perfect foresight approach, where equilibrium prices are the same as their predictions by agents. Agents do not review the prices during their life, although there are no constructive mechanisms to exactly determine the terminal prices and the equilibria may exist for any set of terminal prices (Proposition 1). Existence of equilibria is based on agents' beliefs regarding terminal prices. There is an interesting variation of the model in which some (terminal) prices tend to infinity (Proposition 2). As it turned out, it might be profitable for all consumers to believe that they must abandon all their debts simultaneously if certain state of the environment occurs. This situation is called "default".

We hope that the model considered here may justify both inclusion of liquidity constraints in the theoretical models and adding money as an argument of the utility function [11] to determine prices. This approach could result in extension of standard Arrow–Debreu constructions by introduction of some emission institute like a bank.

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Appendix A. Existence of proper equilibria

As we mentioned above the proof of existence of the proper equilibria is standard. So we only outline the proof.

The proper equilibrium may be viewed as a fixed point of some multivalued correspondence F. The values of the correspondence F are optimal choices in a noncooperative game of $|\mathcal{N}|+1$ participants: $|\mathcal{N}|$ consumers and one fictitious "market participant" or "middleman" (this scheme is used, for example, in [3]). In this game each consumer chooses an optimal consumption plans $c_F^n \in Argmax_x\left\{\mathsf{E}^n\left\{\sum_{t=0}^{T} U^n(x^n(e_t))\right\}|x \in \widetilde{B}^n(p\setminus q)\right\}$, where $\widetilde{B}^n(p\setminus q)$ is a bounded consumption set: $\left\{c^n \in \mathbb{R}_+^{|\mathcal{E}|}|p(s)c^n(s) \leqslant p(s)w^n(s), c^n(e_t) \leqslant \sum_{n\in\mathcal{N}} w^n(e_t), p_T = q\right\}$. The middleman adjusts the prices in order to maximize the profit: $p_F \in Argmax_z\left\{E^{n*}\left\{\sum_{t=0}^{T} z(e_t)(\sum_{n\in\mathcal{N}} c^n(e_t) - \sum_{n\in\mathcal{N}} w^n(e_t))\right\}|z \in Q, z_T = q\right\}$, where Q is a hypercube $[0,P]^{|\mathcal{E}|}$. Middleman's expectations coincide with the expectations of one of the consumers $n^* \in \mathcal{N}$. Thus, F is a convex-valued correspondence from $\widetilde{B}^n(p\setminus q)\times Q$ to $\widetilde{B}^n(p\setminus q)\times Q$. The bounded consumption set $\widetilde{B}^n(p\setminus q)$ depends continuously on $p\setminus q$ (by a Hausdorff metric). As a consequence, the graph of the correspondence F is closed. From here one may conclude that fixed point exists according to Kakutani's theorem. Moreover, the fixed point remains a fixed point if one omits the boundedness condition on the domain $\widetilde{B}^n(p\setminus q)\times Q$.

Appendix B. Existence of equilibria with infinite prices

It is possible to prove Proposition 2 both in terms of potential and actual infinity. Potential infinity is connected with classic mathematical analysis and such notions as "sequence" and "limit". Here we provide the proof of Proposition 2 in terms of actual infinity and non-standard analysis [12]. One can see more details about application of non-standard to mathematical economics in [13].

According to non-standard analysis, one may consider the statement of Proposition 1 (from the Section 2) as a sentence. Here the term "sentence" is understood in the narrow sense of non-standard analysis [12]. Proposition 2 is equivalent to the following sentence:

$$\begin{aligned}&\forall p_T'\forall w^{\mathcal{N}}\Big\{\forall n\in\mathcal{N}\big[(p_T'>0)\cap(w^n>0)\cap(\partial U^n(0)=\infty)\cap(\partial U^n(x)>0)\cap(\partial^2U^n(x)<0)\big]\\
&\Rightarrow\left[\exists(p,c^{\mathcal{N}})\left[\forall n\in\mathcal{N}\forall x^n\begin{pmatrix}(x^n\geqslant 0)\cap(\forall e_T\in\mathcal{E}\sum\limits_{e_t|e_t\leqslant e_T}p(e_t)(w^n(e_t)-x^n(e_t))\geqslant 0)\\ \Rightarrow \mathsf{E}^n\left\{\sum\limits_{t=0}^{T}U^n(c^n(e_t))\right\}\geqslant \mathsf{E}^n\left\{\sum\limits_{t=0}^{T}U^n(x^n(e_t))\right\}\end{pmatrix}\right]\right.\\
&\left.\cap\left[\forall e_T\in\mathcal{E}\sum_{e_t|e_t\leqslant e_T}p(e_t)(w^n(e_t)-c^n(e_t))\geqslant 0\right]\cap\left[\forall e_t\in\mathcal{E}\sum_{n\in\mathcal{N}}w^n(e_t)\geqslant\sum_{n\in\mathcal{N}}c^n(e_t)\right]\cap\left[p_T'=p_T\right]\cap[p>0]\right]\Big\}\end{aligned}\tag{B.1}$$

The sentence (B.1) is a complex combination of notations of mathematical logic. It is well known that even an inequality $\geqslant$ is a notation of ordering function $od(x,y)$ defined on the pairs of the real (or hyperreal) numbers. For example, $\partial U^n(0)=\infty$ is a notation of the sentence $\forall k\in\mathbb{N}\exists\delta>0\forall 0<x<\delta\left(\frac{U^n(x)-U^n(0)}{x}>k\right)$. In the same manner $\partial U^n(x)>0, \mathsf{E}, \sum$, etc. are notations to some sentences.

The sentence (B.1) is closed because it does not contain any parameter that may have an effect on verity of the sentence. We apply the transfer principle to the sentence (B.1). Since the sentence (B.1) holds true for real numbers, it also holds true for hyperreal numbers:

$$\begin{aligned}&\forall^* p_T'\forall^* w^{\mathcal{N}}\left\{\forall n\in\mathcal{N}\begin{bmatrix}(^*p_T'>0)\cap(^*w^n>0)\cap(^*\partial U^n(0)=\infty)\\ \cap(^*\partial U^n(x)>0)\cap(^*\partial^2U^n(x)<0)\end{bmatrix}\right.\\
&\Rightarrow\left[\exists(^*p,^*c^{\mathcal{N}})\left[\forall n\in\mathcal{N}\forall^* x^n\begin{pmatrix}(^*x^n\geqslant 0)\cap(\forall e_T\in\mathcal{E}\sum\limits_{e_t|e_t\leqslant e_T}{}^*p(e_t)(^*w^n(e_t)-{}^*x^n(e_t))\geqslant 0)\\ \Rightarrow \mathsf{E}^n\left\{\sum\limits_{t=0}^{T}{}^*U^n(^*c^n(e_t))\right\}\geqslant \mathsf{E}^n\left\{\sum\limits_{t=0}^{T}{}^*U^n(^*x^n(e_t))\right\}\end{pmatrix}\right]\right.\\
&\left.\left.\cap\left[\forall e_T\in\mathcal{E}\sum_{e_t|e_t\leqslant e_T}{}^*p(e_t)(^*w^n(e_t)-{}^*c^n(e_t))\geqslant 0\right]\cap\left[\forall e_t\in\mathcal{E}\sum_{n\in\mathcal{N}}{}^*w^n(e_t)\geqslant\sum_{n\in\mathcal{N}}{}^*c^n(e_t)\right]\cap\left[^*p_T'={}^*p_T\right]\cap[^*p>0]\right]\right\}.\end{aligned}\tag{B.2}$$

Variables $^*p$, $^*w^n$, $^*c^n$ belong to the set of hyperreal numbers $^*\mathbb{R}$.

We will consider the case of (B.2) in which the endowments $^*w^{\mathcal{N}}$ are real numbers: $^*w^{\mathcal{N}} = st(^*w^{\mathcal{N}}) = w^{\mathcal{N}}$. Using this, we will construct an improper equilibrium $(\widetilde{\mathcal{P}}, \tilde{c}^{\mathcal{N}})$.

Since the material balances $\sum_{n\in\mathcal{N}} w^n(e_t) \geqslant \sum_{n\in\mathcal{N}} {}^*c^n(e_t),\ e_t \in \mathcal{E}$ hold true and the consumption is non-negative, then the consumption ${}^*c^n(e_t)$ is near standard. We define by consumption $\tilde{c}^{\mathcal{N}}$ a standard part of ${}^*c^{\mathcal{N}} : \tilde{c}^n(e_t) = st({}^*c^n(e_t)),\ n \in \mathcal{N},\ e_t \in \mathcal{E}$.

The hierarchic price $\widetilde{\mathcal{P}}$ has the following form:

$$\widetilde{P}(e'_{t'}, e''_{t''}) = \begin{cases} st({}^*p(e'_{t'})/{}^*p(e''_{t''})), & \text{if } {}^*p(e'_{t'})/{}^*p(e''_{t''}) \text{ is near standard}, \\ \infty, & \text{otherwise}. \end{cases}$$

If all fractions in identity $\frac{{}^*p(e'_{t'})}{{}^*p(e''_{t''})} \frac{{}^*p(e''_{t''})}{{}^*p(e'''_{t'''})} \frac{{}^*p(e'''_{t'''})}{{}^*p(e'_{t'})} = 1$ are near standard, then we obtain (4.1) by taking standard parts. If some fraction, for example ${}^*p(e'_{t'})/{}^*p(e''_{t''})$, is not near standard, then $\widetilde{P}(e'_{t'}, e''_{t''}) = \infty$. At the same time, another fraction, for example ${}^*p(e''_{t''})/{}^*p(e'''_{t'''})$, must be infinitesimal, and $\widetilde{P}(e''_{t''}, e'''_{t'''}) = 0$. Thus, (4.1) holds true by definition.

For every complete history $s \in S$ and the corresponding budget constraint $\sum_{e_t|e_t\leqslant e_T} {}^*p(e_t)(w^n(e_t) - {}^*c^n(e_t)) \geqslant 0$ from (B.2) we now are able to determine the maximal price ${}^*p(e_{t'}) = \max_{e_t|e_t\leqslant e_T} {}^*p(e_t)$. By dividing the budget constraint by ${}^*p(e_{t'})$ and taking standard part we obtain a budget constraint from the consumption set (4.2): $\sum_{e_t|e_t\leqslant e_T} \widetilde{P}(e_t, e_{t'})(w^n(e_t) - \tilde{c}^n(e_t)) \geqslant 0$. Therefore, $\tilde{c}^n$ is feasible.

Also $\tilde{c}^n$ is the most preferred on the consumption set (4.2) with respect to preference of the consumer with index n. To show this, suppose that $\tilde{c}^n$ is not the most preferred on the consumption set. Then there exists a real vector x^n such that $E^n\left\{\sum_{t=0}^T U^n(x^n(e_t))\right\} - E^n\left\{\sum_{t=0}^T U^n(\tilde{c}^n(e_t))\right\} = \delta > 0$. There also exists an element ${}^*x^n$ from the hyperreal consumption set (see (B.2)) such that $x^n = st({}^*x^n)$. Thus, $\mathrm{E}^n\left\{\sum_{t=0}^T U^n({}^*x^n(e_t))\right\} - \mathrm{E}^n\left\{\sum_{t=0}^T U^n({}^*c^n(e_t))\right\} > \delta/2$. So ${}^*c^n$ is not the most preferred on the hyperreal budget set. This contradicts (B.2). □

The proof of Proposition 2 with application of potential infinity is a bit difficult. It requires, for example, to construct the sequence of the terminal prices $(p_T)_k$ that tends to the limit. After taking the limit one would have to prove a nontrivial fact that the limiting consumption $(c^n)_k,\ k \to \infty$ is the best on the limit of consumption set.

Although application of non-standard analysis to economic theory is known [13], this approach is still far from being widespread. For this reason the author will be most grateful for any comments if there are any missing moments in the proofs.

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